Unique normal forms for area preserving maps near a fixed point with neutral multipliers

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Abstract

We study normal forms for families of area-preserving maps which have a fixed point with neutral multipliers ± 1 at $\varepsilon = 0$. Our study covers both the orientation-preserving and orientation-reversing cases. In these cases Birkhoff normal forms do not provide a substantial simplification of the system. In the paper we prove that the Takens normal form vector field can be substantially simplified. We also show that if certain non-degeneracy conditions are satisfied no further simplification is generically possible since the constructed normal forms are unique. In particular, we provide a full system of formal invariants with respect to formal coordinate changes.

1   Introduction

The theory of Birkhoff normal forms provides a powerful tool for studying the local dynamics (see e.g. [1, 8]). The normal form theory uses changes of variables to transform a map or a differential equation into a simpler one called a normal form. It is well known that the change of variables is usually not unique and the freedom in its choice can be used to achieve further simplifications. In this paper we consider several problems where these additional simplifications play the central role in the analysis of the normal form and, under certain non-degeneracy conditions, construct unique normal forms.

Similar results for flows on the plane were obtained by various authors following the paper by Baider and Sanders [2]. In this paper we follow a rather classical approach to the problem. In particular, we follow Takens [10] in the approach to normal forms for maps, the simplification procedure is based on the notion of a linear grading function introduced by Kokubu et al. [7]. Finally, we use Lie series for constructing canonical changes of variables which are preferable [4] as it generates formulae which are easier for implementations.

We note that our procedure is based on a simplification of the Birkhoff normal form. The characteristic property of the Birkhoff normal is a symmetry generated by the linear part of the original system at $\varepsilon = 0$. Therefore our results can be directly applied to systems which posses the same symmetries independently of their origin.

In the paper all theorems are stated for families of maps, but the assumptions of these theorems can be satisfied by an individual map F_0 which can be obviously considered as a family which is constant as a function of ε. Therefore all theorems can be used to make conclusions about F_0 by setting $\varepsilon = 0$ in the corresponding equations.

1.1   Orientation preserving families

Let $F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic family of area and orientation preserving maps which depend on a small parameter ε. Assume that at $\varepsilon = 0$ the origin is a fixed point:

$$F_0(0) = 0.$$

Since F_0 is area and orientation preserving $\det DF_0(0) = 1$. Hence we can denote the two eigenvalues of the Jacobian matrix $DF_0(0)$ by μ_0 and μ_0^{-1}. These eigenvalues are often called *multipliers* of the fixed point. The Birkhoff normal form depends on the values of the multipliers.

If μ_0 is real and $|\mu_0| \neq 1$, the fixed point is hyperbolic and Moser proved that the Birkhoff normal form is unique and analytic [9]. If μ_0 is not real, the fixed point is elliptic (obviously $|\mu_0| = 1$ in this case). If the elliptic fixed point is not resonant, the Birkhoff normal form is unique. In the resonant case, the unique normal form was recently obtained by the authors [5].

In this paper we study the case of a parabolic fixed point with $\mu_0 = \mu_0^{-1} = \pm 1$. Though the word "parabolic" is often used for a generic fixed point with $\mu_0 = 1$ only. If the Jacobian matrix is not diagonalisable there is an area-preserving linear change of coordinates (*i.e.*, defined by a matrix with the unit determinant), after which we have

$$DF_0(0) = \begin{pmatrix} \mu_0 & a \\ 0 & \mu_0 \end{pmatrix}$$

with $a = \pm 1$. A reflection in the vertical axes changes the sign of a. Therefore without loosing in generality we assume

$$DF_0(0) = \mu_0 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If $\mu_0 = 1$ the classical theory of Birkhoff normal forms does not provide any simplification for the map since all terms of the Taylor expansion are "resonant". On the other hand, it is well known that F_ε can be formally interpolated by an autonomous Hamiltonian flow, i.e., there is a formal Hamiltonian such that its time-one map coincides with the Taylor series of the map:

$$F_\varepsilon = \Phi^1_{h_\varepsilon},$$

where $\Phi^1_{h_\varepsilon}$ is the time-one map generated by the Hamiltonian h_ε. In particular h_ε is a formal integral of the map: $h_\varepsilon \circ F_\varepsilon = h_\varepsilon$. Generically the series h_ε diverge [6]. The formal interpolating Hamiltonian can be substantially simplified.

Theorem 1 *Let F_ε be an analytic (or C^∞, or formal) family of area preserving maps defined in a neighbourhood of the origin such that $F_0(0) = 0$, $DF_0(0)$ has a double eigenvalue $\mu_0 = 1$ and is not diagonalisable. Then there exists a canonical[1] formal change of variables such that*

$$F_\varepsilon = \phi_\varepsilon^{-1} \circ \Phi^1_{h_\varepsilon} \circ \phi_\varepsilon$$

with

$$h_\varepsilon(x, y) = \frac{y^2}{2} + U_\varepsilon(x), \tag{1}$$

where

$$U_\varepsilon(x) = \sum_{k+3m \geq 3} u_{km} x^k \varepsilon^m$$

is a formal series in two variables.

This form of the interpolating Hamiltonian is very convenient for studying bifurcations. Indeed, a truncation of the series h_ε provides a good approximation

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[1] Possibly reflecting the orientation

for the map. On the other hand, the phase portrait of a Hamiltonian written in the form (1) is easy to analyse.

Under an additional non-degeneracy assumption, a further simplification of h_ε is possible and leads to the unique normal form. The coefficients of the unique normal form are invariants of the map under formal (and consequently analytic) changes of variables. It is interesting to note that the unique normal form still contains infinitely many coefficients. Consequently the number of independent formal invariants is infinite.

Theorem 2 *Let F_ε satisfy the assumptions of Theorem 1. If there is $n \geq 3$ such that $u_{k0} = 0$ for all $k < n$ and $b := u_{n0} \neq 0$, then there exists a canonical formal change of variables such that h_ε takes the form (1) with*

$$U_\varepsilon(x) = bx^n + \sum_{\substack{k+nm>n \\ k \neq -1 \mod n}} u_{km} x^k \varepsilon^m.$$

The coefficients of the series are defined uniquely by the map F_ε.

We note that for $\varepsilon = 0$ the theorem follows from the previous one in combination with the result by Baider and Sanders [2] who proved that a planar Hamiltonian with a nilpotent singularity has the following unique normal form

$$h_0 = \frac{y^2}{2} + \sum_{\substack{k \geq n \\ k \neq -1 \mod n}} b_k x^k,$$

where n corresponds to the lowest non-vanishing order, i.e., $b_n \neq 0$.

In the case of $\mu_0 = -1$ the Birkhoff normal form is odd in (x, y) and the Takens normal form vector field is described by an even Hamiltonian.

Theorem 3 *Let F_ε be an analytic (or C^∞, or formal) family of area-preserving maps defined in a neighbourhood of the origin such that $F_0(0) = 0$, $DF_0(0)$ has a double eigenvalue $\mu_0 = -1$ and is not diagonalisable. Then there exists a canonical formal change of variables such that*

$$F_\varepsilon = -\phi_\varepsilon^{-1} \circ \Phi_{h_\varepsilon}^1 \circ \phi_\varepsilon,$$

where

$$h_\varepsilon(x, y) = \frac{y^2}{2} + U_\varepsilon(x) \qquad with \qquad U_\varepsilon(x) = \sum_{k+2m \geq 2} u_{km} x^{2k} \varepsilon^m.$$

Moreover, the coefficients of the series are defined uniquely by the family F_ε.

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It is interesting to note that in this case the uniqueness of the coefficients does not require any additional non-degeneracy condition.

Finally we consider a family with diagonalisable $DF_0(0)$.

Theorem 4 *Let F_ε be an analytic (or C^∞, or formal) family of area preserving maps defined in a neighbourhood of the origin such that $F_0(0) = 0$, $DF_0(0)$ has a double eigenvalue $\mu_0 = 1$ or $\mu_0 = -1$ and is diagonalisable and a non-degeneracy condition is satisfied. Then there exists a canonical formal change of variables ϕ_ε such that*

$$F_\varepsilon = \mu_0 \, \phi_\varepsilon^{-1} \circ \Phi^1_{h_\varepsilon} \circ \phi_\varepsilon$$

with

$$h_\varepsilon(x, y) = xy^2 + ax^3 + A(x, \varepsilon) + yB(xy^2, \varepsilon), \tag{2}$$

where A and B are formal series of the form

$$A(x, \varepsilon) = \sum_{k+3m \geq 4} a_{km} x^k \varepsilon^m, \qquad B(xy^2, \varepsilon) = \sum_{k+m \geq 1} b_{km} (xy^2)^k \varepsilon^m. \tag{3}$$

If $\mu_0 = -1$ the formal Hamiltonian h_ε is even.

Note that this theorem does not include the uniqueness statement. We will prove that the coefficients of the Hamiltonian h_ε are invariant under formal tangent-to-identity canonical changes which preserve the form of the series. Unlike the previous theorems the uniqueness of the normal form does not follow in the case of a negative a because there are two linear changes of variables which do not change the form of the leading part of the series but may affect higher orders. As a result there are possibly three different h_ε corresponding to a single family F_ε.

1.2 Orientation reversing families

Now let us assume that $F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the area but reverses the orientation. Similar to the previous subsection we assume that at $\varepsilon = 0$ the origin is a fixed point:

$$F_0(0) = 0.$$

Since $\det DF_0(0) = -1$, we can denote the multipliers of the origin by μ_0 and $-\mu_0^{-1}$. Since F_0 is real, both multiplies are necessarily real. We expect that in the hyperbolic case a result similar to [9] should be valid. In this paper we consider the parabolic case which corresponds to $\mu_0 = -1$. Then there is a canonical change of coordinates which diagonalises the Jacobian and we can assume without loosing in generality:

$$DF_0(0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

According to the classical normal form theory, the Hamiltonian of the Takens normal form vector field is an odd function in x, i.e., $h_\varepsilon(-x, y) = -h_\varepsilon(x, y)$. The formal series h_0 starts with cubic terms, therefore in the leading order

$$h_0(x, y) = bxy^2 + ax^3 \,.$$

This Hamiltonian has the same form as in the orientation-preserving case with a diagonalisable Jacobian. This similarity leads to the similarity in the structure of the normal form.

Theorem 5 *Let F_0 be an analytic (or C^∞, or formal) area-preserving orientation-reversing map defined in a neighbourhood of the origin such that $F_0(0) = 0$, $DF_0(0)$ has eigenvalues -1 and 1, and the leading order of the normal form is non-degenerate, i.e, $b \neq 0$. Then there exists a canonical formal change of variables such that*

$$F_\varepsilon = \mathrm{Diag}\,(-1, 1)\, \phi_0^{-1} \circ \Phi_{h_0}^1 \circ \phi_0,$$

where

$$h_0(x, y) = xy^2 + ax^3 + A(x, \varepsilon) + yB(xy^2, \varepsilon)\,,$$

where A and B are formal series of the form

$$A(x, \varepsilon) = \sum_{k+3m \geq 4} a_{mk} x^{2k-1} \varepsilon^m\,, \qquad B(xy^2, \varepsilon) = \sum_{k+m \geq 1} b_{mk} (xy^2)^{2k-1} \varepsilon^m\,.$$

The coefficients of the series are defined uniquely by the map.

Structure of the paper

In Section 2 we describe the usage of formal series and quasi-homogeneous polynomials. In Section 3 we prove a theorem about formal interpolation of a map by an autonomous Hamiltonian. In Section 4 we derive the unique normal form for the case $\mu_0 = 1$ and non-diagonalisable $DF_0(0)$. In Section 5 the Birkhoff normal form is derived. In Section 6 we study the case $\mu_0 = -1$. The case of diagonalisable Jacobian is considered in Section 7. Finally, in Section 8 we consider orientation reversing families.

The results of Sections 3 and 5 are used in the proof of the main theorems and are included for completeness of the arguments.

2 Formal series and quasi-homogeneous polynomials

In this paper we will be mainly interested in transformations given in the form of formal power series. We will consider the series in powers of the space variables

(x, y) and the parameter ε. The series have the form

$$g(x, y, \varepsilon) = \sum_{k,l,m} c_{klm} x^k y^l \varepsilon^m \, .$$

A formal series is treated as a collection of coefficients. Formal series form an infinite dimensional vector space. The addition, multiplication, integration and differentiation are defined in a way compatible with the common definition on the subset of convergent series.

As the series involve several variables, it is convenient to group terms which are "of the same order". A usual choice is to consider x, y and ε to be of the same order. But for purpose of this paper it is much more convenient to assume

- x is of order k_0,

- y is of order l_0,

- ε is of order m_0,

where k_0, l_0, m_0 will be chosen later. Then a monomial $x^k y^l \varepsilon^m$ is considered to be of order $kk_0 + ll_0 + mm_0$ (this is a grading function in terminology of [7]). We can write

$$g(x, y, \varepsilon) = \sum_{p \geq 0} g_p(x, y, \varepsilon)$$

where

$$g_p(x, y, \varepsilon) = \sum_{kk_0 + ll_0 + mm_0 = p} c_{klm} x^k y^l \varepsilon^m$$

is a quasi-homogeneous polynomial of order p since it has the property

$$g_p(\lambda^{k_0} x, \lambda^{l_0} y, \lambda^{m_0} \varepsilon) = \lambda^p g_p(x, y, \varepsilon)$$

for any $\lambda \in \mathbb{R}$. We stress that this notation does not refer to a resummation of the divergent series but simply indicates the order in which the coefficients of the series are to be treated.

In order to give a rigorous background for manipulation with formal series we remind that the space of formal series $\mathfrak{H}$ can be considered as an infinite dimensional vector space equipped with the following metric. Let g and $\tilde{g}$ be two formal series and let p denote the lowest (quasi-homogeneous) order of $g - \tilde{g}$. If $g \neq \tilde{g}$ then p is finite and

$$d(g, \tilde{g}) = 2^{-p},$$

otherwise

$$d(g, g) = 0 \, .$$

It can be check that $(\mathfrak{H}, d)$ is a complete metric space. Moreover polynomials are dense in $(\mathfrak{H}, d)$. Hence we can define formal convergence and formal continuity on

the space of formal series. In particular an operator is formally continuous if each coefficients of a series in its image is a function of a finite number of coefficients of a series in its argument.

We note that any of the series involved in next definitions may diverge.

Let χ and g be two formal power series. The linear operator defined by the formula

$$L_\chi g = \{g, \chi\} \tag{4}$$

is called *the Lie derivative* generated by χ. We note that if χ starts with an order p and g starts with an order q, then the series $L_\chi g$ starts with the order $p + q - (k_0 + l_0)$ as the Poisson bracket involves differentiation with respect to x and y.

If

$$p \geq k_0 + l_0 + 1 \tag{5}$$

the lowest order in $L_\chi g$ is at least $q + 1$. Then we define the exponent of L_χ by

$$\exp(L_\chi)g = \sum_{k \geq 0} \frac{1}{k!} L_\chi^k g \,, \tag{6}$$

where L_χ^k stands for the operator L_χ applied k times. The lowest order in the series $L_\chi^k g$ is at least $q + k$. So the series (6) converges with respect to the metric d, i.e., each coefficient of the result depends on a finite number of coefficients of the series χ and g.

We consider the formal series

$$\Phi_\chi^1(x, y) = \big(\exp(L_\chi)x,\ \exp(L_\chi)y\big).$$

We say that Φ_χ^1 is a Lie series generated by the formal Hamiltonian χ. If χ is polynomial the series converge on a poly-disk and coincide with a map which shifts points along trajectories of the Hamiltonian system with Hamiltonian function χ. For this reason we will call Φ_χ^1 a time-one map of the Hamiltonian χ even in the case when the series do not converge.

We note that it is easy to construct the formal series for the inverse map:

$$\Phi_\chi^{-1}(x, y) = \big(\exp(-L_\chi)x,\ \exp(-L_\chi)y\big).$$

Then $\Phi_\chi^1 \circ \Phi_\chi^{-1}(x, y) = (x, y)$. We also note that

$$g \circ \Phi_\chi^1 = \exp(L_\chi)g \,.$$

These formula are well known for convergent series and can be extended onto $\mathfrak{H}$ due to the density property.

3   Formal interpolation

In this section it will be convenient to set $k_0 = 2$, $l_0 = 3$ and $m_0 = 6$. So we will order terms in power series supposing that

$$x \quad \text{is of order 2,}$$
$$y \quad \text{is of order 3,}$$
$$\varepsilon \quad \text{is of order 6.}$$

Next theorem states that F_ε can be formally interpolated by an autonomous Hamiltonian flow.

Theorem 6 *Let F_ε be a family of area-preserving maps such that $F_0(0) = 0$ and*

$$DF_0(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then there exists a unique (up to adding a formal series in ε only) formal Hamiltonian

$$h_\varepsilon(x, y) = \sum_{p \geq 6} h_p(x, y, \varepsilon), \qquad \text{where} \quad h_p(x, y, \varepsilon) = \sum_{2k+3l+6m=p} h_{klm} x^k y^l \varepsilon^m \,,$$

such that

$$\Phi^1_{h_\varepsilon} = F_\varepsilon. \tag{7}$$

Moreover, if F_ε is odd in (x, y) then h_ε is even.

Before proving the theorem we need a simple technical statement.

Lemma 7 *Let f_p and g_{p+1} be quasi-homogeneous polynomial of orders p and $p+1$ respectively. There is a quasi-homogeneous polynomial h_{p+3} of order $p+3$ which satisfies the system*

$$\begin{aligned} \partial_y h_{p+3} &= f_p \\ -\partial_x h_{p+3} &= g_{p+1} \end{aligned} \tag{8}$$

if and only if

$$\text{div} \begin{pmatrix} f_p \\ g_{p+1} \end{pmatrix} = 0 \,. \tag{9}$$

If exists, the polynomial h_{p+3} is unique in the class of quasi-homogeneous polynomials (up to adding $h_{00m} \varepsilon^m$). Moreover, if f_p and g_{p+1} are odd in (x, y) then h_{p+3} is even.

Proof. The polynomials can be written in the form:

$$f_p = \sum_{2k+3l+6m=p} a_{klm} x^k y^l \varepsilon^m \,,$$

$$g_{p+1} = \sum_{2k+3l+6m=p+1} b_{klm} x^k y^l \varepsilon^m \,, \tag{10}$$

$$h_{p+3} = \sum_{2k+3l+6m=p+3} h_{klm} x^k y^l \varepsilon^m \,.$$

Substituting these polynomials into the equations and collecting similar terms we see that (9) is equivalent to

$$k a_{k,l-1,m} + l b_{k-1,l,m} = 0, \quad k,l \geq 0 \tag{11}$$

and (8) is equivalent to

$$h_{klm} = \frac{a_{k,l-1,m}}{l} \quad \text{for } l > 0 \quad \text{and} \quad h_{klm} = -\frac{b_{k-1,l,m}}{k} \quad \text{for } k > 0 \,. \tag{12}$$

Equalities (12) are compatible if and only if (11) is satisfied. We note that the equations do not involve the coefficient h_{00m} which can be chosen arbitrarily.

If f_p and g_{p+1} are odd in (x,y), i.e. $a_{klm} = 0$ and $b_{klm} = 0$ for $k+l$ even, then $h_{klm} = 0$ for $k + (l-1)$ even (or equivalently for $k+l$ odd), i.e. h is even. ∎

Proof of Theorem 6.   The Taylor series for $F_\varepsilon : (x,y) \mapsto (x_1, y_1)$ can be written as a sum of quasi-homogeneous polynomials:

$$\begin{cases} x_1 = x + y + \sum\limits_{p \geq 4} f_p(x,y,\varepsilon), \\ y_1 = y + cx^2 + \sum\limits_{p \geq 4} g_{p+1}(x,y,\varepsilon), \end{cases} \tag{13}$$

where f_p and g_{p+1} are quasi-homogeneous polynomials:

$$\begin{cases} f_p(x,y,\varepsilon) = \sum\limits_{2k+3l+6m=p} f_{klm} x^k y^l \varepsilon^m, \\ g_{p+1}(x,y,\varepsilon) = \sum\limits_{2k+3l+6m=p+1} g_{klm} x^k y^l \varepsilon^m. \end{cases}$$

The time-one map of h_ε is given by the Lie series

$$\Phi_{h_\varepsilon}^1(x,y) = \begin{pmatrix} x + L_{h_\varepsilon} x + \sum\limits_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k x \\ y + L_{h_\varepsilon} y + \sum\limits_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k y \end{pmatrix},$$

where the operator L_{h_ε} is defined by

$$L_{h_\varepsilon}(\varphi) = \{\varphi, h_\varepsilon\} = \frac{\partial \varphi}{\partial x} \frac{\partial h_\varepsilon}{\partial y} - \frac{\partial \varphi}{\partial y} \frac{\partial h_\varepsilon}{\partial x}.$$

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We note that the inequality (5) is satisfied and consequently the formal series is well defined.

Now we use induction to show that there is a formal Hamiltonian h_ε such that the formal series $\Phi^1_{h_\varepsilon}$ coincides with the Taylor expansion of F_ε at all orders. We note the first component of the series starts with the quasi-homogeneous order 2 (i.e. with x) and the second one starts with 3 (i.e. with y). Therefore it is convenient to consider an order p in the first component simultaneously with the order $p+1$ in the second one. In this situation we say that we consider a term of the order $(p, p+1)$. The terms of order $(2, 3)$ in (7) coincide for an arbitrary h_ε.

To make the combinatorics easier we introduce the notation

$$L_s \varphi = \{\varphi, h_{s+5}\}.$$

Then

$$L_{h_\varepsilon} \varphi = \{\varphi, h_\varepsilon\} = \sum_{p \geq 6} \{\varphi, h_p\} = \sum_{s \geq 1} L_s \varphi.$$

Note that L_s maps any quasi-homogeneous polynomial of order j into a quasi-homogeneous polynomial of order $j + s$. So we have

$$
\begin{aligned}
L^k_{h_\varepsilon} \varphi_j &= \left(\sum_{s \geq 1} L_s \right)^k \varphi_j = \sum_{\substack{s_1 + s_2 + \ldots + s_k \geq k \\ s_1, s_2, \ldots, s_k \geq 1}} L_{s_1} \ldots L_{s_k} \varphi_j \\
&= \sum_{m \geq k} \sum_{\substack{s_1 + s_2 + \ldots + s_k = m \\ s_1, s_2, \ldots, s_k \geq 1}} L_{s_1} \ldots L_{s_k} \varphi_j.
\end{aligned}
$$

Now let us consider the components of $\Phi^1_{h_\varepsilon}$ at order $(p, p+1)$. The first component is

$$\left[L_{h_\varepsilon} x + \sum_{k \geq 2} \frac{1}{k!} L^k_{h_\varepsilon} x \right]_p = L_{p-2} x + \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots s_k = p-2} L_{s_1} \ldots L_{s_k} x$$

and the second component

$$\left[L_{h_\varepsilon} y + \sum_{k \geq 2} \frac{1}{k!} L^k_{h_\varepsilon} y \right]_{p+1} = L_{p-2} y + \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots s_k = p-2} L_{s_1} \ldots L_{s_k} y.$$

Since

$$
\begin{aligned}
L_{p-2} x &= \{x, h_{p+3}\}, \\
L_{p-2} y &= \{y, h_{p+3}\}
\end{aligned}
$$

the equation (7) can be rewritten as

$$
\begin{pmatrix} \frac{\partial h_{p+3}}{\partial y} \\ -\frac{\partial h_{p+3}}{\partial x} \end{pmatrix} = \begin{pmatrix} f_p - \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} x \\ g_{p+1} - \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} y \end{pmatrix}. \tag{14}
$$

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First we check that the equations can be solved for $p = 3$. Since

$$
\begin{aligned}
f_3 &= y, \\
g_4 &= cx^2
\end{aligned}
$$

the equations (14) take the form

$$
\frac{\partial h_6}{\partial y} = y, \tag{15}
$$

$$
-\frac{\partial h_6}{\partial x} = cx^2.
$$

These equations can be easily solved:

$$
h_6 = \frac{y^2}{2} - c\frac{x^3}{3}. \tag{16}
$$

Let us now proceed with the induction step. Suppose that for some $p \geq 4$ we have $h_6, \ldots, h_{p+2}$ such that $\left[\Phi^1_{h_\varepsilon}\right]_{(s,s+1)} = [F_\varepsilon]_{(s,s+1)}$ for every $s \leq p-1$ where $h_\varepsilon = \sum_{k \geq 6} h_k$ with arbitrary h_k for $k > p+2$. We want to find h_{p+3} such that $\left[\Phi^1_{h_\varepsilon}\right]_{(s,s+1)} = [F_\varepsilon]_{(s,s+1)}$ for every $s \leq p$ where $h_\varepsilon = \sum_{k \geq 6} h_k$ with arbitrary h_k for $k > p+3$.

The quasi-homogeneous polynomial h_{p+3} is defined from equations (14) uniquely up to a function of ε if and only if the right hand side is divergence free, which follows from the area-preservation property due to the following lemma.

Lemma 8 *If two families of area-preserving maps for which $F_\varepsilon(0) = \tilde{F}_\varepsilon(0) = 0$ and*

$$
DF_\varepsilon(0) = D\tilde{F}_\varepsilon(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
$$

coincide up to the order $(p-1, p)$ then the divergence of the respective maps at the order $(p, p+1)$ are equal.

Proof. We can write the map in the form

$$
F_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + f(x, y, \varepsilon) \\ y + g(x, y, \varepsilon) \end{pmatrix}.
$$

Since F_ε is area-preserving we have

$$
\det DF_\varepsilon = \begin{vmatrix} 1 + \partial_x f & 1 + \partial_y f \\ \partial_x g & 1 + \partial_y g \end{vmatrix} = 1
$$

which is equivalent to

$$
\partial_x f + \partial_y g + \{f, g\} - \partial_x g \equiv 0. \tag{17}
$$

Similar identities are valid for $\tilde{F}_\varepsilon$. We use the tilde to distinguish between the maps. We have $f_k = \tilde{f}_k$ and $g_{k+1} = \tilde{g}_{k+1}$ for $k \leq p-1$. We use the area-preserving property (17) and collect the terms of order $p-2$

$$\partial_x f_p + \partial_y g_{p+1} - \partial_x g_p + \sum_{\substack{k+l=p+3 \\ k,l \geq 4}} \{f_k, g_l\} = 0,$$

which we reorder as

$$\partial_x f_p + \partial_y g_{p+1} = \partial_x g_p - \sum_{\substack{k+l=p+3 \\ k,l \geq 4}} \{f_k, g_l\}$$

$$= \partial_x \tilde{f}_p + \partial_y \tilde{g}_{p+1}.$$

So the divergences are equal. ■

We finish the proof by the following observation. The right hand side of (14) depends on h_k with $k \leq p+2$. Let $\tilde{h}_\varepsilon = \sum_{k=6}^{p+2} h_k$. The maps $\Phi^1_{\tilde{h}_\varepsilon}$ and F_ε satisfy the assumptions of the previous lemma therefore their orders $(p, p+1)$ have the same divergence. That is the solvability condition for the equation (14). The even property of h_ε for odd F_ε follows from Lemma 7. ■

Now we will proof the interpolation theorem for a family with diagonalisable Jacobian.

Theorem 9 Let $F_\varepsilon = \begin{pmatrix} f_\varepsilon(x, y) \\ g_\varepsilon(x, y) \end{pmatrix}$ be a family of area-preserving maps such that $F_0(0) = 0$ and

$$DF_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then there exists a unique (up to adding a formal series in powers ε only) formal Hamiltonian

$$h_\varepsilon(x, y) = \sum_{p \geq 3} h_p(x, y, \varepsilon), \qquad \text{where} \quad h_p(x, y, \varepsilon) = \sum_{k+l+3m=p} h_{klm} x^k y^l \varepsilon^m,$$

such that
$$\Phi^1_{h_\varepsilon} = F_\varepsilon. \tag{18}$$

Moreover, if f_ε is odd in x and g_ε is even in x then h_ε is odd in x.

Proof. We assume that

$$\begin{array}{ll} x & \text{is of order 1,} \\ y & \text{is of order 1,} \\ \varepsilon & \text{is of order 3.} \end{array}$$

The rest of the proof is similar to Theorem 6 so we briefly describe the necessary modifications.

We can write the Taylor series for $F_\varepsilon : (x, y) \mapsto (x_1, y_1)$ as a sum of quasi-homogeneous polynomials:

$$\begin{cases} x_1 = x + \sum_{p \geq 2} f_p(x, y, \varepsilon), \\ y_1 = y + \sum_{p \geq 2} g_p(x, y, \varepsilon), \end{cases}$$

where f_p and g_p are quasi-homogeneous polynomials:

$$\begin{cases} f_p(x, y, \varepsilon) = \sum_{k+l+3m=p} f_{klm} x^k y^l \varepsilon^m, \\ g_p(x, y, \varepsilon) = \sum_{k+l+3m=p} g_{klm} x^k y^l \varepsilon^m. \end{cases}$$

In Lemma 7 instead of (8) we have

$$\begin{aligned} \partial_y h_{p+1} &= f_p \\ -\partial_x h_{p+1} &= g_p \end{aligned}$$

instead of (9) we have

$$\mathrm{div} \begin{pmatrix} f_p \\ g_p \end{pmatrix} = 0 \,.$$

If f_p is odd in x and g_p is even in x then h_{p+1} is odd in x.

Since

$$F_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + f(x, y, \varepsilon) \\ y + g(x, y, \varepsilon) \end{pmatrix}.$$

instead of (17) of Lemma 8 we have

$$\mathrm{div} \begin{pmatrix} f \\ g \end{pmatrix} + \{f, g\} = 0,$$

which is used to check the solvability condition for h_p. On the first step h_3 should be find in the form of a third order polynomial in x and y which depends depends on f_2 and g_2.

In the rest the proof is essentially the same as in Theorem 6. ∎

4 Simplified normal forms: $\mu_0 = 1$ with the non-diagonalisable linear part.

In this section we show that the formal Hamiltonian constructed in the previous section can be substantially simplified by a formal canonical change of variables leading to the unique normal form. This results extends earlier results of [2] onto an one parametric unfolding of the nilpotent Hamiltonian singularity.

Similar to the previous section we write h as a formal sum of quasi-homogeneous polynomials.

Proposition 10 *If $h(x, y, \varepsilon) = \frac{y^2}{2} + bx^3 + \sum_{p \geq 7} h_p(x, y, \varepsilon)$ where*

$$h_p(x, y, \varepsilon) = \sum_{2k+3l+6m=p} h_{klm} x^k y^l \varepsilon^m$$

then there exists a formal canonical substitution such that the Hamiltonian takes the form

$$\widetilde{h}(x, y, \varepsilon) = \frac{y^2}{2} + bx^3 + \sum_{k+3m \geq 4} u_{km} x^k \varepsilon^m. \tag{19}$$

Proof. We prove the theorem by induction. The leading order of the Hamiltonian has the form

$$h_6(x, y, \varepsilon) = \frac{y^2}{2} + bx^3. \tag{20}$$

Suppose we transformed the Hamiltonian to the desired form up to the order p with $p \geq 6$. We will find a quasi-homogeneous polynomial χ_p such that $\widetilde{h} = h \circ \Phi_{\chi_p}^1$ has the desired form up to the order $p + 1$.

For any quasi-homogeneous polynomial χ_p of order p we have

$$\begin{aligned}
\widetilde{h} &= \exp(L_{\chi_p} h) = h + \{h, \chi_p\} + \widehat{O}_{2p-4} \\
&= h + \{h_6, \chi_p\} + \widehat{O}_{p+2},
\end{aligned}$$

where $\widehat{O}_k$ denotes a formal series which starts with a quasi-homogeneous polynomial of order k. We see that this change does not affect any term of order less or equal p. Collecting all terms of order $p + 1$ we get

$$\widetilde{h}_{p+1} = h_{p+1} + \{h_6, \chi_p\}. \tag{21}$$

Let $p + 1 = 3q + r$ with $r \in \{0, 1, 2\}$. The largest possible power of y in h_{p+1} does not exceed q and we can write

$$h_{p+1} = \sum_{\substack{0 \leq l \leq q \\ (3l+r) \text{ even}}} y^{q-l} u_{r+3l}(x, \varepsilon) \quad \text{and} \quad \widetilde{h}_{p+1} = \sum_{\substack{0 \leq l \leq q \\ (3l+r) \text{ even}}} y^{q-l} \widetilde{u}_{r+3l}(x, \varepsilon),$$

$$\chi_p = \sum_{\substack{0 \leq l \leq q \\ (3l+r-1) \text{ even}}} y^{q-l} v_{r+3l-1}(x, \varepsilon), \tag{22}$$

where u_j, $\widetilde{u}_j$ and v_j are quasi-homogeneous polynomials in x and ε of order j. In order to simplify our notation we allow v_j with $j < 0$ assuming that $v_j = 0$ if $j < 0$.

Taking into account

$$\{h_6, \chi_p\} = 3bx^2 \frac{\partial \chi_p}{\partial y} - y \frac{\partial \chi_p}{\partial x}$$

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and collecting terms of the same order in y we get (excluding the y-independent terms) from (21):

$$\tilde{u}_{r+3l} = u_{r+3l} + 3b(q+1-l)x^2 v_{r+3l-4} - \frac{\partial v_{r+3l+2}}{\partial x}, \quad 0 \le l \le q-1. \qquad (23)$$

We choose v_{r+3l+2} such that $\tilde{u}_{r+3l} = 0$.

If $p+1 = 3q+r$ is odd then the last equation is at y^1 and v_{r+3q-1} is defined such that $\tilde{u}_{r+3q-3} = 0$ and so, $\tilde{h}_{p+1} = 0$.

If $p+1$ is even then the last equality is at y^0:

$$\tilde{u}_{3q+r} = u_{3q+r} + 3bx^2 v_{3q+r-4}, \qquad (24)$$

where $v_{3q+r-4} = v_{p-3}$ was defined from (23) on the previous step with $l = q-1$.

We have got $\tilde{u}_j = u_j = 0$ except $\tilde{u}_{p+1}$. So, we can construct χ_p such that $\tilde{h}_{p+1}(x,y,\varepsilon) = \tilde{u}_{p+1}(x,\varepsilon)$ is independent from y. $\blacksquare$

Theorem 1 follows from Theorem 6, equation (16) and Proposition 10.

Note that further simplification of the Hamiltonian h_ε is possible. Indeed, if in (23) $r+3l+2 = 0 \mod 6$ then v_{r+3l+2} is defined up to const $\varepsilon^{(r+3l+2)/6}$. Therefore in (24) the polynomial v_{p-3} is defined by (23) up to an arbitrary quasi-homogeneous polynomial in (x^3,ε). So, if $b \ne 0$ we can use this freedom to eliminate x^k for $k = 2 \mod 3$ in $\tilde{h}$.

We will prove a more general result:

Proposition 11 *Let*

$$h(x,y,\varepsilon) = \frac{y^2}{2} + bx^n + \sum_{k+nm>n} c_{km} x^k \varepsilon^m$$

with $b \ne 0$, $n \ge 3$. Then there exists a formal canonical change of variables such that the Hamiltonian takes the form

$$\widetilde{h}(x,y,\varepsilon) = \frac{y^2}{2} + bx^n + \sum_{\substack{k+nm \ge n \\ k \ne -1 \mod n}} u_{km} x^k \varepsilon^m. \qquad (25)$$

The coefficients of the series are defined uniquely. If h is even in (x,y) then $\widetilde{h}$ is also even in (x,y).

Proof. We will use the following ordering of variables:

- x is of order 2;

- y is of order n;

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- ε is of order 2n.

Then h can be written as $h = \sum_{p \geq 2n} h_p$, where

$$h_{2n} = \frac{y^2}{2} + bx^n,$$

$$h_p = \sum_{2k+2nm=p} c_{km} x^k \varepsilon^m, \quad p > 2n.$$

Let χ_p be a quasi-homogeneous polynomial of order $p \geq n + 4$. After the change of variables given by $\Phi_{\chi_p}^1$ the Hamiltonian takes the form

$$\widetilde{h} = \exp(L_{\chi_p})h = h + \{h, \chi_p\} + \widehat{O}_{2p-4} = h + \{h_{2n}, \chi_p\} + \widehat{O}_{p+n-1} + \widehat{O}_{2p-4}.$$

Consequently $\widetilde{h}_k = h_k$ for $k < p + n - 2$ and

$$\widetilde{h}_{p+n-2} = h_{p+n-2} + \{h_{2n}, \chi_p\}. \tag{26}$$

We will consider the homological operator

$$L(\chi_p) = \{h_{2n}, \chi_p\} = nbx^{n-1}\frac{\partial \chi_p}{\partial y} - y\frac{\partial \chi_p}{\partial x}. \tag{27}$$

Let $p + n - 2 = qn + r$ with $r \in \{0, 1, \ldots, n-1\}$. Then $\widetilde{h}_{p+n-2}$ and h_{p+n-2} can be written as

$$\widetilde{h}_{p+n-2} = \sum_{0 \leq l \leq q} y^l \widetilde{u}_{n(q-l)+r}(x, \varepsilon), \quad h_{p+n-2} = \sum_{0 \leq l \leq q} y^l u_{n(q-l)+r}(x, \varepsilon),$$

where $\widetilde{u}_j$ and u_j are quasi-homogeneous polynomials in (x, ε) of order j. The highest power of y in χ_p does not exceed

$$\left[\frac{p}{n}\right] = \begin{cases} q-1 & \text{if } r \in \{0, 1, \ldots, n-3\}, \\ q & \text{if } r \in \{n-2, n-1\}. \end{cases}$$

Moreover if $r = n - 1$ then $p = qn + 1$ and χ_p does not contain the term proportional to y^q because the orders of x and ε are larger or equal 2. So,

$$\chi_p = \delta_{r,n-2}y^q v_0 + \sum_{0 \leq l \leq q-1} y^l v_{p-nl}(x, \varepsilon), \tag{28}$$

where $\delta_{r,n-2}$ is Kronecker symbol. Calculating $\frac{\partial \chi_p}{\partial y}$ and $\frac{\partial \chi_p}{\partial x}$ and substituting them into (27) we get

$$\begin{aligned}
L(\chi_p) &= \sum_{0 \leq l \leq q-2} nb(l+1)y^l x^{n-1} v_{p-n(l+1)}(x, \varepsilon) \\
&\quad + \delta_{r,n-2} nbq y^{q-1} v_0 x^{n-1} - \sum_{1 \leq l \leq q} y^l \frac{\partial v_{p-n(l-1)}}{\partial x}(x, \varepsilon).
\end{aligned}$$

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Substituting $L(\chi_p)$ into (26) and considering coefficients of y^l we get

$$l = q: \qquad \widetilde{u}_r = u_r - \frac{\partial v_{r+2}}{\partial x}, \tag{29}$$

$$l = q - 1: \qquad \widetilde{u}_{n+r} = u_{n+r} + \delta_{r,n-2}nbqv_0 x^{n-1} - \frac{\partial v_{n+r+2}}{\partial x}, \tag{30}$$

$$1 \le l \le q - 2: \qquad \widetilde{u}_{n(q-l)+r} = u_{n(q-l)+r} \tag{31}$$

$$+ nb(l+1)x^{n-1}v_{n(q-l-2)+r+2} - \frac{\partial v_{n(q-l)+r+2}}{\partial x},$$

$$l = 0: \qquad \widetilde{u}_{p+n-2} = u_{p+n-2} + nbx^{n-1}v_{p-n}. \tag{32}$$

We can choose v_{r+2} in (29) such that $\widetilde{u}_r = 0$. Similarly we can choose v_{n+r+2} in (30) such that $\widetilde{u}_{n+r} = 0$. If $r \ne n - 2$ then v_{n+r+2} is unique.

The equations (29)–(31) with $\widetilde{u}_j = 0$ can be considered as two recurrent systems of equations on $v_{r+2+2kn}$ and on $v_{r+2+n+2kn}$ respectively. If $r \ne n - 2$ the polynomial v_{p-n} is defined uniquely from (31) with $l = 2$ and, consequently, the polynomial $\widetilde{u}_{p+n-2}$ in (32) is also defined uniquely.

If $r = n-2$ then in (30) we can take an arbitrary v_0 and $v_{n+r+2} = v_{2n}$ contains const ε. So, the polynomial $v_{p-n} = v_{q(n-1)}$ is defined up to an arbitrary quasi-homogeneous polynomial of order $p - n = n(q - 1)$ in (x^n, ε). Consequently, by choosing this polynomial in (32) all terms containing $x^{kn+(n-1)}$ in $\widetilde{u}_{p+n-2}$ can be eliminated.

We note that if in (29)–(32) u_j are even functions of x then v_{j+2} are odd. And if n is even then x^{n-1} is odd and $x^{n-1}v_{j+2}$ are even. So, $\widetilde{u}_{p+n-2}$ are even and $\widetilde{h}$ is even too.

In order to complete the proof we need to establish uniqueness of the series (25). Let $\mathcal{H}_p$ denote the set of all quasi-homogeneous polynomials of order p. Then $L : \mathcal{H}_p \to \mathcal{H}_{p+n-2}$. We have chosen χ_p on each step such that $\widetilde{h}_{p+n-2}$ is in the complement to the range of L.

Let us consider the kernel of L. Substituting (28) into equation

$$L(\chi_p) = 0$$

and considering the coefficients in front of y^l we get equations (29)–(32) with $\widetilde{u}_j = u_j = 0$. It's not difficult to see that $\dim \ker L = k$ if $p = 2nk$ otherwise $\ker L$ is trivial. Since $L(h_{2n}^j) = 0$ the $\ker L$ consists of $\varepsilon^{k-j}h_{2n}^j$ only.

Suppose there exist two Hamiltonians $\widetilde{h}$ and $\widetilde{h}'$ of the form (25). Then there exists $\chi = \sum_{k \ge p} \chi_k$ such that $\widetilde{h}' = \exp(L_\chi)\widetilde{h}$. Then $\widetilde{h}'_k = \widetilde{h}_k$ for $k < p + n - 2$ and

$$\widetilde{h}'_{p+n-2} = \widetilde{h}_{p+n-2} + L(\chi_p).$$

Since both $\widetilde{h}'_{p+n-2}$ and $\widetilde{h}_{p+n-2}$ are in the complement of the range of L, $\widetilde{h}'_{p+n-2} = \widetilde{h}_{p+n-2}$ and $L(\chi_p) = 0$. Consequently, $\chi_p = 0$ if $p \ne 0 \mod 2n$ or $\chi_p = $

$\sum_{1 \le j \le k} \alpha_j \varepsilon^{k-j} h_{2n}^j$ if $p = 2nk$. It means that

$$\chi = \sum_{1 \le j \le k} \alpha_j \varepsilon^{k-j} h_{2n}^j + \sum_{j \ge p+1} \chi_j$$

and the $\tilde{h}'$ and $\tilde{h}$ are also connected by $\tilde{\chi} = \sum_{j \ge p+1} \chi_j$. The order of $\tilde{\chi}$ is at least $p+1$ and we can repeat the arguments to show that $\tilde{h}'_{p+2} = \tilde{h}_{p+2}$ and so on. We conclude by induction that $\tilde{h}'$ coincides with $\tilde{h}$ at all orders. ■

Theorem 2 follows from Theorem 1 and Proposition 11.

5 Birkhoff normal form for $\mu_0 = -1$

Now we consider the case $\mu_0 = -1$. As in the case of $\mu_0 = 1$ the Jacobian of F_0 has a double eigenvalue and can be non-diagonalisable. The following theorem follows from the classical results and is included for the sake of completeness.

Theorem 12 *If $F_\varepsilon(x, y)$ is a family of area-preserving maps such that $F_0(0) = 0$ and $DF_0(0) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, then there exists a canonical formal coordinate change ϕ such that*

$$\widetilde{F}_\varepsilon = \phi^{-1} \circ F_\varepsilon \circ \phi$$

is an odd series in the space variables.

Proof. We will use the following ordering:

$$\begin{aligned} x & \quad \text{is of order 2,} \\ y & \quad \text{is of order 3,} \\ \varepsilon & \quad \text{is of order 6.} \end{aligned}$$

The proof goes by induction, we normalise the orders $(p-3, p-2)$ consecutively using quasi-homogeneous Hamiltonian flows

$$\chi_p(x, y, \varepsilon) = \sum_{2k+3l+6m=p} c_{klm} x^k y^l \varepsilon^m.$$

The corresponding Lie series are well defined for $p \ge 6$. Now consider the change of variables

$$\widetilde{F}_\varepsilon = \Phi_{\chi_p}^{-1} \circ F_\varepsilon \circ \Phi_{\chi_p}^1, \tag{33}$$

we will use the polynomial χ_p to eliminate even terms at the order $(p-3, p-2)$. Equation (33) is equivalent to

$$\Phi_{\chi_p}^1 \circ \widetilde{F}_\varepsilon = F_\varepsilon \circ \Phi_{\chi_p}^1. \tag{34}$$

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We have

$$
\begin{aligned}
F_\varepsilon \circ \Phi^1_{\chi_p} &= F_\varepsilon + \{F_\varepsilon, \chi_p\} + \widehat{O}_{(2p-8, 2p-7)} \\
&= F_\varepsilon + \begin{pmatrix} -\frac{\partial \chi_p}{\partial y} \\ \frac{\partial \chi_p}{\partial x} \end{pmatrix} + \widehat{O}_{(p-2, p-1)},
\end{aligned}
$$

where $\widehat{O}_j$ denotes a series which starts with a quasi-homogeneous order at least j or higher. Taking into account the form of the map we get

$$
F_\varepsilon \circ \Phi^1_{\chi_p}(x, y) = \begin{pmatrix} -x - y + f_\varepsilon(x, y) - \frac{\partial \chi_p}{\partial y}(x, y) + \widehat{O}_{p-2} \\ -y + g_\varepsilon(x, y) + \frac{\partial \chi_p}{\partial x}(x, y) + \widehat{O}_{p-1} \end{pmatrix}. \tag{35}
$$

The time-one map is given by

$$
\Phi^1_{\chi_p}(x, y) = \begin{pmatrix} x + \frac{\partial \chi_p}{\partial y} + \widehat{O}_{2p-8} \\ y - \frac{\partial \chi_p}{\partial x} + \widehat{O}_{2p-7} \end{pmatrix}.
$$

Suppose

$$
\widetilde{F}_\varepsilon(x, y) = \begin{pmatrix} -x - y + \widetilde{f}_\varepsilon(x, y) \\ -y + \widetilde{g}_\varepsilon(x, y) \end{pmatrix}.
$$

Then we have

$$
\Phi^1_{\chi_p} \circ \widetilde{F}_\varepsilon(x, y) = \begin{pmatrix} -x - y + \widetilde{f}_\varepsilon(x, y) + \frac{\partial \chi_p}{\partial y}\Big|_{(-x, -y)} + \widehat{O}_{p-2} \\ -y + \widetilde{g}_\varepsilon(x, y) - \frac{\partial \chi_p}{\partial x}\Big|_{(-x, -y)} + \widehat{O}_{p-1} \end{pmatrix}.
$$

Substituting this and (35) in (34) and collecting the terms of the order $(p-3, p-2)$ we get equations for χ_p:

$$
\begin{cases}
\left[\widetilde{f}_\varepsilon(x, y)\right]_{p-3} = [f_\varepsilon(x, y)]_{p-3} - \frac{\partial \chi_p}{\partial y}(x, y) - \frac{\partial \chi_p}{\partial y}(-x, -y), \\
[\widetilde{g}_\varepsilon(x, y)]_{p-2} = [g_\varepsilon(x, y)]_{p-2} + \frac{\partial \chi_p}{\partial x}(x, y) + \frac{\partial \chi_p}{\partial x}(-x, -y).
\end{cases}
$$

We see in the equation that all even terms of $\partial_y \chi_p$ and $\partial_x \chi_p$ are doubled and all odd terms are cancelled. Let us now choose χ_p such that all even terms of $\widetilde{f}_\varepsilon$ and $\widetilde{g}_\varepsilon$ are eliminated. Let χ_p be an odd polynomial such that

$$
\begin{cases}
\frac{\partial \chi_p}{\partial y} = \frac{1}{2} [f_\varepsilon(x, y)]^{even}_{p-3}, \\
-\frac{\partial \chi_p}{\partial x} = \frac{1}{2} [g_\varepsilon(x, y)]^{even}_{p-2},
\end{cases} \tag{36}
$$

then $\left[\widetilde{f}_\varepsilon(x, y)\right]_{p-3}$ and $[\widetilde{g}_\varepsilon(x, y)]_{p-2}$ are odd. A sufficient condition for existence of χ_p is zero divergence which follows from the area-preservation property of F_ε.

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Suppose all orders of F_ε up to $(p-4, p-3)$ are odd. Since F_ε is area-preserving we have

$$\det DF_\varepsilon = \begin{vmatrix} -1 + \partial_x f_\varepsilon & -1 + \partial_y f_\varepsilon \\ \partial_x g_\varepsilon & -1 + \partial_y g_\varepsilon \end{vmatrix} = 1$$

which is equivalent to

$$-(\partial_x f_\varepsilon + \partial_y g_\varepsilon) + \{f_\varepsilon, g_\varepsilon\} + \partial_x g_\varepsilon \equiv 0. \tag{37}$$

Taking order $p-5$ we get

$$-\frac{\partial}{\partial x} f_{p-3} - \frac{\partial}{\partial y} g_{p-2} + \sum_{\substack{k+l-5=p-5 \\ k,l \geq 4}} \{f_k, g_l\} + \frac{\partial}{\partial x} g_{p-3} = 0.$$

Note that the last two terms are even, hence we conclude that

$$\frac{\partial f_{p-3}^{even}}{\partial x} + \frac{\partial g_{p-2}^{even}}{\partial y} = 0.$$

Hence (36) has a solution χ_p.

We have proved that the even part of the order $(p-3, p-2)$ with $p \geq 6$ can be eliminated provided all previous orders are odd. To complete the proof we need a base for the induction. We note that the expansion of the map starts with the order $(2, 3)$. There are no even terms of order 3 or lower (the only term of order 2 is x, and the only term of order 3 is y). Thereby the induction base is established. ∎

6 Unique normal form for $\mu_0 = -1$

Now we consider $\mu_0 = -1$. We have

$$F_\varepsilon(x, y) = \begin{pmatrix} -x - y + f(x, y, \varepsilon) \\ -y + g(x, y, \varepsilon) \end{pmatrix} = -\Phi_h^1(x, y).$$

In the case $\mu_0 = -1$ the family F_ε can be transformed to the Birkhoff normal form. Then F_ε is odd (Theorem 12) and $h(x, y, \varepsilon) = \sum_{p \geq 4} h_p(x, y, \varepsilon)$ is an even function on the space variables (Theorem 6). Then due to Proposition 11 there exists a formal canonical substitution such that the Hamiltonian takes the form

$$\widetilde{h}(x, y, \varepsilon) = \frac{y^2}{2} + \sum_{k+m \geq 2} u_{km} x^{2k} \varepsilon^m$$

and the coefficients of the series are defined uniquely. Theorem 3 is proved.

7 Diagonalisable Jacobian

We will use the following ordering:

$$\begin{aligned} x & \quad \text{is of order 1,} \\ y & \quad \text{is of order 1,} \\ \varepsilon & \quad \text{is of order 3.} \end{aligned}$$

By Theorem 9 there exists a Hamiltonian $h_\varepsilon(x,y) = h_3 + \sum_{p \geq 4} h_p$ such that $F = \Phi_h^1$. The polynomial of the third order h_3 depends only on (x,y). After a linear change it can be written as $h_3 = \alpha x^3 + \beta xy^2$ (it is convenient to look for this change in two steps: first make a substitution $(x,y) \mapsto (x+cy,y)$ and then $(x,y) \mapsto (x,cx+y)$ swapping x and y using $(x,y) \mapsto (y,-x)$ if necessary).

Proposition 13 *If a formal Hamiltonian has the form*

$$h(x,y,\varepsilon) = ax^3 + xy^2 + \sum_{k+l+3m \geq 4} h_{klm} x^k y^l \varepsilon^m \,,$$

then there is a formal tangent-to-identity canonical change of variables such that it is transformed into

$$\widetilde{h}(x,y,\varepsilon) = ax^3 + xy^2 + A(x,\varepsilon) + yB(xy^2,\varepsilon) \tag{38}$$

where

$$A(x,\varepsilon) = \sum_{k+3m \geq 4} a_{km} x^k \varepsilon^m \,, \qquad B(xy^2,\varepsilon) = \sum_{k+m \geq 1} b_{km} (xy^2)^k \varepsilon^m \,. \tag{39}$$

The coefficients of the series are defined uniquely. If h is odd in x then $\widetilde{h}$ is also odd in x.

Proof. We will use χ_p for simplifying h_{p+1}. Collecting all terms of order $p+1$ we get

$$\widetilde{h}_{p+1} = h_{p+1} + \{h_3, \chi_p\} \,. \tag{40}$$

Let

$$\chi_p = \sum_{l=0}^p y^{p-l} v_l(x,\varepsilon),$$

$$h_{p+1} = \sum_{l=0}^{p+1} y^{p+1-l} u_l(x,\varepsilon), \quad \widetilde{h}_{p+1} = \sum_{l=0}^{p+1} y^{p+1-l} \widetilde{u}_l(x,\varepsilon),$$

where v_l, u_l and $\widetilde{u}_l$ are quasi-homogeneous polynomials in x and ε of order l. Substituting $\widetilde{h}_{p+1}$, h_{p+1} and χ_p into (40) we get $p+2$ equalities (for $l = 0, 1, ..., p, p+1$):

$$
\begin{align}
\widetilde{u}_0 &= u_0 + pv_0, \tag{41}\\
\widetilde{u}_1 &= u_1 + [(p-1)v_1 - 2xv_1'], \tag{42}\\
\widetilde{u}_l &= u_l + 3a(p-l+2)x^2v_{l-2} + [(p-l)v_l - 2xv_l'], \quad 2 \leq l \leq p-1, \tag{43}\\
\widetilde{u}_p &= u_p + 6ax^2v_{p-2} - 2xv_p', \tag{44}\\
\widetilde{u}_{p+1} &= u_{p+1} + 3ax^2v_{p-1}. \tag{45}
\end{align}
$$

Set $v_0 = -\frac{u_0}{p}$ to make $\widetilde{u}_0 = 0$. Let

$$v_l(x, \varepsilon) = \sum_{m=0}^{[l/3]} \nu_m x^{l-3m} \varepsilon^m,$$

$$u_l(x, \varepsilon) = \sum_{m=0}^{[l/3]} v_m x^{l-3m} \varepsilon^m,$$

$$\widetilde{u}_l(x, \varepsilon) = \sum_{m=0}^{[l/3]} \widetilde{v}_m x^{l-3m} \varepsilon^m.$$

We will choose ν_m such that $\widetilde{u}_l$ takes the simplest form, i.e. $\widetilde{v}_m$ vanishes if possible. Equations (42)–(43) for v_l can be written as

$$2xv_l' - (p-l)v_l = f_l(x, \varepsilon), \qquad 1 \leq l \leq p-1,$$

where $f_l(x, \varepsilon) = u_l - 3a(p-l+2)x^2v_{l-2}$, $v_{-1} = 0$.
  Let $f_l(x, \varepsilon) = \sum_{m=0}^{[l/3]} \varphi_m x^{l-3m} \varepsilon^m$. Then for $0 \leq m \leq [l/3 - 1/3]$

$$\nu_m [3l - 6m - p] = \varphi_m$$

If l, m, p are such that $3l - 6m - p \neq 0$ then ν_m can be chosen such that $\widetilde{v}_m = 0$. But if $3l - 6m - p = 0$ then we cannot change $\widetilde{v}_m$ by choosing ν_m. It takes place if $p = 3s$, $l = s+2m$, i.e. h and $\widetilde{h}$ have the same coefficients in front of $y^{2k+1}x^k\varepsilon^m$ (for $k \geq 1$, $m \geq 0$).
  In (44) we put $v_p(x, \varepsilon) = \sum_{m=0}^{[p/3]} \nu_m x^{p-3m} \varepsilon^m$. Then

$$xv_p'(x, \varepsilon) = \sum_{m=0}^{[p/3]} (p - 3m)\nu_m x^{p-3m} \varepsilon^m.$$

If $p - 3m \neq 0$ we can chose ν_m such that the corresponding terms in $\widetilde{u}_p$ vanish.
  If $p - 3m = 0$ then in $\widetilde{u}_p$ does not vanish the term with $x^0\varepsilon^{p/3}$, i.e. in $\widetilde{h}_{p+1}$ does not vanish the term with $y\varepsilon^{p/3}$. In (45) v_{p-1} is already defined from (44) with $l = p - 1$. Therefore $\widetilde{u}_{p+1}$ does not vanish in general case.

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So, $\widetilde{h}_{p+1}$ contains terms with $x^k \varepsilon^m$ and $y(xy^2)^k \varepsilon^m$.

The uniqueness follows from the usual analysis of the complement to the range of the homological operator based on the equations (41)–(45) with $u_l = \widetilde{u}_l = 0$.
∎

8   Orientation reversing families

Now we will consider the area-preserving orientation-reversing families.

Theorem 14 *If $F_\varepsilon(x,y)$ is a family of area-preserving maps such that $F_0(0) = 0$ and $DF_0(0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then there exists a canonical formal coordinate change ϕ such that*

$$\widetilde{F}_\varepsilon = \phi^{-1} \circ F_\varepsilon \circ \phi = \begin{pmatrix} f(x,y,\varepsilon) \\ g(x,y,\varepsilon) \end{pmatrix},$$

where f is odd and g is even in x.

Proof. We will use the following ordering:

$$
\begin{aligned}
x & \quad \text{is of order 1,} \\
y & \quad \text{is of order 1,} \\
\varepsilon & \quad \text{is of order 3.}
\end{aligned}
$$

In the rest of the proof is completely analogous to the proof of Theorem 12.   ∎

We can write the map in the form

$$F_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + \sum_{p \geq 2} f_p(x,y,\varepsilon) \\ y + \sum_{p \geq 2} g_p(x,y,\varepsilon) \end{pmatrix},$$

where f_p is odd and g_p is even in x, therefore h is odd in x (a proof is similar to Theorem 9), then $h_3 = ax^3 + bxy^2$. By Proposition 13 there exists a formal canonical change of space variables such that the Hamiltonian takes the form

$$\widetilde{h}(x,y,\varepsilon) = ax^3 + xy^2 + \sum_{k+3m \geq 4} a_{mk} \varepsilon^m x^{2k-1} + y \sum_{k+m \geq 1} b_{mk} \varepsilon^m (xy^2)^{2k-1}$$

and Theorem 4 follows.

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