Complex Continued Fractions and Extremal Theory

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Abstract

For each of the five Euclidean rings of quadratic integers, we consider a complex continued fraction algorithm with digits in the ring. We show for each algorithm that the maximal digit obeys a Fréchet distribution. We use this to find a limiting distribution for cusp excursions on Bianchi orbifolds associated with the aforementioned rings of quadratic integers.

1 Introduction

The goal of the first part of this manuscript is to generalise an extreme value theorem for continued fraction expansions of real numbers to the complex case. In general a continued fraction is an expression of the form

\[ z = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}. \]  

We denote the right hand side by \([a_0, a_1, a_2, \ldots].\) In the more familiar case of regular continued fractions where \(a_n \in \mathbb{N}\) and \(z \in (0, \infty),\) the choice of the digits \(a_n\) in (1) is unique. For this choice of algorithm, we have the following result due to Galambos [Gal72].

Proposition 1.1 (Galambos). For regular continued fractions, if \(y > 0,\) then

\[ \lim_{N \to \infty} \nu \left\{ z \in [0, 1] : \max_{1 \leq n \leq N} a_n \leq \frac{yN}{\log 2} \right\} = e^{-1/y}, \]

where \(d\nu = (\log 2)^{-1}d\lambda(z)/(1 + z)\) is the Gauss measure on \([0, 1].\)

We obtain an analogous result for the case \(z \in \mathbb{C}\) as follows. If \(d = 1, 2, 3, 7\) or 11, we can consider expressions of the form (1) with \(a_n \in \mathfrak{o}_d:\) the ring of integers of the field extension \(\mathbb{Q}(\sqrt{-d}).\) We generate such an expansion by requiring for all \(n > 0\) that the norm of the tail

\[ [0, a_n, a_{n+1}, a_{n+2}, \ldots] \]  

is as small as possible. See Section 2 for more details. We call this the nearest-integer \(\mathfrak{o}_d\)-continued fraction expansion of \(z.\) For \(d = 1,\)
this is more commonly known as the Hurwitz complex continued fraction expansion, first described in [Hur87]. Let $K_1$ be the square in $\mathbb{C}$ consisting of all points with real and imaginary parts in $[-1/2, 1/2]$. By [Ei+19], there exists a Borel probability measure $\mu_1$ on $K_1$ equivalent to the Lebesgue measure for which the digits of the Hurwitz continued fraction expansion form a stationary sequence. With respect to this measure, we have the following result [Kir21].

**Proposition 1.2** (Kirsebom). There exists a constant $C_1 > 0$ such that for any $y > 0$, we have that

$$\lim_{N \to \infty} \mu_1 \left\{ z \in K_1 : \max_{1 \leq n \leq N} |a_n(z)| \leq C_1 y \sqrt{N} \right\} = e^{-1/y^2},$$

where $(a_n(z))_n$ are the digits in the Hurwitz complex continued fraction expansion of $z$.

This distribution is a special case of the Fréchet Distribution, which is a family of distributions with cumulative distribution function $0 < y \mapsto e^{-y^\alpha}$ for some $\alpha > 0$. The proof of Proposition 1.2 relies on several properties of the Hurwitz continued fraction algorithm discovered in [Ei+19]. Based on recent work by Ei, Nakada and Natsui in [ENN23], we extend Kirsebom’s result to other values of $d$.

**Theorem 1.1** (Fréchet Law for the Maximal Digit). Let $d = 1, 2, 3, 7, 11$. Let $K_d$ be the set of all points in $\mathbb{C}$ for which the closest element of $\mathfrak{o}_d$ is 0. There exists a constant $C_d > 0$ depending on $d$ such that for any $y > 0$, we have that

$$\lim_{N \to \infty} \eta \left\{ z \in K_d : \max_{1 \leq n \leq N} |a_n(z)| \leq C_d y \sqrt{N} \right\} = e^{-1/y^2},$$

where $(a_n(z))_n$ are the digits in the nearest-integer $\mathfrak{o}_d$-continued fraction expansion of $z$ and $\eta$ is any probability measure that is absolutely continuous with respect to the Lebesgue measure. In particular, $C_d$ does not depend on $\eta$.

The goal of the second part of this manuscript is to use Theorem 1.1 to study cusp excursions of the geodesic flow on certain three-dimensional analogues of the modular surface, which date back to Bianchi [Bia92]. We recall that the group of orientation-preserving isometries of the three-dimensional hyperbolic space $H^3$ can be identified with the projective special linear group $\text{PSL}_2(\mathbb{C})$, see Section 4. We call the subgroup $\text{PSL}_2(\mathfrak{o}_d)$ consisting of all the matrices in $\text{PSL}_2(\mathbb{C})$ with entries in $\mathfrak{o}_d$ a Bianchi group. The quotient space $P_d = \text{PSL}_2(\mathfrak{o}_d) \backslash H^3$ is called a Bianchi orbifold. We denote the unit tangent bundle of $P_d$, i.e. the submanifold of the tangent bundle consisting of vectors of unit length, by $T^1P_d$.

**Theorem 1.2.** Let $P_d$ be a Bianchi orbifold corresponding to the imaginary quadratic field extension $\mathbb{Q}[\sqrt{-d}]$ for $d = 1, 2, 3, 7, 11$. For a vector $v \in T^1P_d$, let $v(t)$ be the vector obtained by applying the geodesic
There exists some constant $\alpha > 0$ such that
\[
\lim_{T \to \infty} m \left\{ v \in T^1P_d : \sup_{0 \leq t \leq T} d(v, v(t)) - \frac{1}{2} \log T \leq \log(y) + \alpha \right\} = e^{-1/y^2},
\]
where $m$ is the Liouville measure, i.e. the natural volume on $T^1P_d$.

The theorem above is a higher-dimensional analogue of the following result due to the second author [Pol09]. Let $M$ be the modular surface and let $m_M$ be the Liouville measure for its unit tangent bundle $T^1M$.

**Proposition 1.3** ([Pol09]). Let $y > 0$. Then
\[
\lim_{T \to \infty} m_M \left\{ v \in T^1M : \sup_{0 \leq t \leq T} d(v, v(t)) - \frac{1}{2} \log \frac{3y}{\pi} \right\} = e^{-1/y}.
\]

These results are refinements of Sullivan’s logarithm law [Sul82] for the aforementioned Bianchi orbifolds and the modular surface.

**Proposition 1.4** (Sullivan). Let $\Gamma$ be discrete subgroup of isometries of the $n$-dimensional hyperbolic space $H^n$. Assume $\Gamma \setminus H^n$ is of finite volume but not compact, then
\[
\limsup_{t \to \infty} \frac{d(v, v(t))}{\log t} = \frac{1}{d-1}
\]
for almost all vectors $v$ with respect to the Liouville measure in the unit tangent bundle of $\Gamma \setminus H^n$.

## 2 Complex Continued Fractions

Given any continued fraction expansion $[a_0, a_1, \cdots]$, we define the associated quotient pair of sequences $(p_n)_n$ and $(q_n)_n$ by the recursive equations
\[
\begin{align*}
p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2} \\
q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2}.
\end{align*}
\]

It is well-known that for all $n$,
\[
p_{n-1}q_n - p_nq_{n-1} = (-1)^n
\]
and
\[
\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n].
\]

We call $p_n/q_n$ the $n$-th convergent of the continued fraction.

The rest of his section is devoted to describing the nearest-integer $\omega_d$-continued fraction expansion algorithm in detail. Recall that the ring of integers $\mathcal{O}_d$ is defined as the set of all $\mathbb{Q}[(\sqrt{-d})]$-valued solutions to the quadratic equations of the form $z^2 + bz + c$, where $b, c \in \mathbb{Z}$. If $d$ is a square-free integer, then
\[
\mathcal{O}_d = \mathbb{Z}[\omega] = \{ n + m\omega : n, m \in \mathbb{Z} \},
\]
where \( \omega = \left( \frac{-1 + \sqrt{-d}}{2} \right) \) if \( d = 3 \mod 4 \) and \( \omega = \sqrt{-d} \) otherwise.

We defined the set \( K_d \subset \mathbb{C} \) in Theorem 1.1 as
\[
K_d = \{ z \in \mathbb{C} : |z| < |z - \lambda| \text{ for all } \lambda \in \mathfrak{o}_d \}.
\]

In other words, \( K_d \) is the Dirichlet fundamental domain centred at the origin for translations by elements in \( \mathfrak{o}_d \). Explicitly, this is given by
\[
K_d := \left\{ x + iy : x \in \mathbb{C} : |x| < \frac{1}{2}, |y| < \frac{\sqrt{d}}{2} \right\} \text{ if } d = 1, 2, \text{ and }
\]
\[
K_d := \left\{ x + iy : x \in \mathbb{C} : |x| < \frac{1}{2}, \right\}
\]
\[
|x - y\sqrt{d}| < \frac{d + 1}{4}, |x + y\sqrt{d}| < \frac{d + 1}{4} \right\} \text{ if } d = 3, 7, 11.
\]

See [ENN22] for more details and for complex continued fraction algorithms associated to other choices of fundamental domains.

**Definition 2.1.** Let \( d = 1, 2, 3, 7, 11 \). Define the function \( \lfloor \cdot \rfloor_d : \mathbb{C} \setminus \{0\} \to \mathfrak{o}_d \) such that \( \lfloor z \rfloor_d = a \) implies that \( z \in a + K_d \).

**Remark 2.1.** The function \( z \mapsto \lfloor z \rfloor_d \) is not uniquely determined if \( z \in a + \partial K_d \) for some \( a \in \mathfrak{o}_d \). Any choice will work, however, since we will only consider properties which hold almost everywhere with respect to the Lebesgue measure.

We associate to the function \( \lfloor \cdot \rfloor_d \) the complex Gauss map
\[
G : K_d \to K_d : z \mapsto \frac{1}{z - \lfloor 1 \rfloor_d}.
\]

Define the sequences \((z_n)_n\) and \((a_n)\) by
\[
z_0 = z, a_0(z) = \lfloor z \rfloor_d,
\]
\[
z_n = \frac{1}{G^{n-1}(z - a_0(z))}, a_n(z) = \lfloor z \rfloor_d \text{ for } n \geq 1.
\]

If the expression \([a_0, a_1, \ldots]\) converges, then \(G^{n-1}(z - a_0(z))\) is equal to the tail \([2]\). Since the latter is in \( K_d \), the above algorithm indeed minimizes the euclidean norm of the tails. We note that by definition of the Dirichlet fundamental domain, the integer \( \lfloor z \rfloor_d \) is the element of \( \mathfrak{o}_d \) with the nearest Euclidean distance to \( z \), whence the name of the algorithm.

**Proposition 2.1.** The continued fraction \([a_0, a_1, \ldots]\) with \( a_i \) chosen by \([6]\), converges for all \( z \in \mathbb{C} \). It stops in a finite number of steps if and only if \( z \in \mathbb{Q}[\sqrt{-d}] \). If \( p_n/q_n \) is the \( n \)-th convergent of \( z \), then
\[
\left| \frac{q_{n-1}}{q_n} \right| < 1
\]
for all $n$. The identity

$$q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1] \quad (8)$$

also holds.

Equation (7) was proved by Hurwitz in [Hur87] for $d = 1, 3$. A discussion of this fact for all algorithms under consideration can be found in [Lak73]. Equation (8) is a standard result and can be proven by induction. In general such a reversed-digit continued fraction expansion cannot be obtained with the nearest-integer algorithm. In the case $d = 1$, it is shown in [Ei+19] that there is a dual continued fraction expansion associated to these reversed digits.

Remark 2.2. The values $d = 1, 2, 3, 7, 11$ are those for which a Euclidean division algorithm exists for $\sigma_d$ with respect to the partial order given by the norm. For other values of $d$, the nearest-integer algorithm doesn’t converge. This is discussed in e.g. [ENN22].

From this point on, when we refer to the continued fraction expansion of a given complex number, we exclusively mean with respect to the nearest-integer $\sigma_d$-continued fraction algorithm for $d = 1, 2, 3, 7, 11$.

3 Extremal Value Theory for the Digits of Continued Fractions.

The goal of this section is to prove Theorem 1.1 for nearest-integer $\sigma_d$-continued fraction expansions. There are two main elements to this proof. We first establish an asymptotic result for the probability that the norm of a given digit exceeds some large value of $t$ in Lemma 3.2. Subsequently, we use a mixing property of the digits to argue that $(|a_n|)_n$ behaves like an independent sequence for the purposes of extreme value theory. Both the asymptotics and mixing results are based on recent work by Hiromi Ei, Hitoshi Nakada and Rie Natsui, who extended their previous results with Shunji Ito for Hurwitz complex continued fractions in [Ei+19] to all complex continued fraction algorithms under our consideration in [ENN23].

The following lemma summarises some results from from [ENN22] and [ENN23] that we require. We assume throughout that $d = 1, 2, 3, 7, 11$.

Lemma 3.1. Let $\lambda$ be the Lebesgue measure on $C$. The map $G$ admits an ergodic invariant measure $\mu_d$ which is absolutely continuous with respect to the Lebesgue measure. Furthermore there exists a $C' > 0$ such that

$$\frac{1}{C'} \lambda(A) \leq \mu_d(A) \leq C' \lambda(A) \quad (9)$$

for every Borel set $A$. In fact, there is a partition

$$K_d = \bigcup_{k=1}^{J} V_k$$

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into path-connected sets whose boundaries consist of a finite union of segments of circles and lines such that there exist Borel sets $V_k^* \subset \{ w \in \mathbb{C} : |w| \geq 1 \}$ of positive measure satisfying

$$d\mu_d = C \left( \sum_{k=1}^t \chi_{V_k} \int_{V_k^*} \frac{1}{|z-w|} \, d\lambda(w) \right) \, d\lambda(z), \quad (10)$$

where $\chi_{V_k}$ is the characteristic function on $V_k$, and $C$ is a normalizing constant which ensures $\mu_d$ is a probability measure.

See Figure 1 for an illustration of this partition and the dual sets $V_k^*$.

**Remark 3.1.** Since the set $V_k^*$ lies outside of the unit disk and \( \sup_{z \in V_k^*} |z| < 1 \), we have that the function

$$g_k(z) := \int_{V_k^*} \frac{1}{|z-w|} \, d\lambda(w)$$

is infinitely differentiable with bounded derivatives when viewed as a function on $\mathbb{R}^2$. Indeed, the function $(z, w) \mapsto |z-w|^4$ is real analytic and bounded outside of a neighbourhood of the diagonal $z = w$. The same holds for all its partial derivatives. The claim is then proved by an application of the Lebesgue dominated convergence theorem.

The following proof extends Lemma 2 in [Kir21] to the nearest-integer $\sigma_d$-continued fractions under consideration.

**Lemma 3.2.** Let $d = 1, 2, 3, 7, 11$. There is a constant $H > 0$ such that

$$\mu_d\{ z \in K_d : |a_1(z)| > t \} = \frac{H}{t^2} + O \left( \frac{1}{t^3} \right) \text{ as } t \to \infty,$$

where $a_1$ is the first digit in the nearest-integer $\sigma_d$-continued fraction.

**Proof.** Let $\iota$ be the inversion map $\hat{\mathbb{C}} \to \hat{\mathbb{C}} : z \mapsto 1/z$ on the Riemann sphere $\mathbb{C} \cup \{ \infty \}$. Define the set $A_t$ for $t > 0$ by setting

$$A_t := \iota(\{ z \in K_d : |a_1(z)| > t \}) = \{ w \in \iota(K_d) : |\lfloor w \rfloor_d | > t \}.$$

Let $D(r)$ be the disc of radius $r$ centred around the origin and denote its compliment by $D(r)^C$. It follows from the definition of $K_d$ that

$$D(t+1)^C \subset A_t \subset D(t-1)^C.$$

For $t > 1$. By applying $\iota$ to the formula above and taking the measure $\mu_d$ we obtain that

$$|\mu_d\{ z \in K_d : |a_1(z)| > t \} - \mu_d(D(1/t))| \leq C' \lambda \left( D \left( \frac{1}{t-1} \right) \setminus D \left( \frac{1}{t+1} \right) \right),$$

where $C'$ is the constant in (9). The Lebesgue measure of the annulus is of order $O(1/t^2)$, so we obtain that

$$\mu_d\{ z \in K_d : |a_1(z)| > t \} = \mu_d(D(1/t)) + O \left( \frac{1}{t^3} \right).$$
(a) The partition $\bigcup_{k=1}^{d'} V_k$ for $d = 2$.

(b) In the $d = 2$ setting from left to right: the set of all points $z$ for which $1/z$ lies in the sets $V_d^*$ and $V_d^{**}$.

Figure 1: From top to bottom, these are Figures 12 and 16 from [ENN23].
It thus suffices to prove that for some $H$,

$$\mu_d(D(1/t)) = \frac{H}{t^2} + O\left(\frac{1}{t^3}\right) \quad \text{as } t \to \infty.$$  

We use the notation of Remark 3.1. By (10), it is enough to show that the functions

$$f_k(x) = \int_{V_k \cap D(x)} \chi_{V_k \cap D(x)}(z)g_k(z)d\lambda(z)$$

satisfy $f_k(\frac{x}{t}) = W_k + O\left(\frac{1}{t}\right)$ for some constant $W_k$. By Taylor’s theorem, it suffices to show that $f_k$ is three times differentiable in $[0, \epsilon)$ for some $\epsilon > 0$. Recall that the boundary of $V_k$ consists of a finite union of circle segments and line segments. For sufficiently small $x$, the disk $D(x)$ intersects two segments if 0 is a vertex of $V_k$. For example, if $d = 2$, this is the case for $V_1, \ldots, V_8$ in Figure 1a. Else $D(x)$ intersects at most one line segment. In any case, these segments admit a smooth parametrisation of the form $r \mapsto re^{i\theta(r)}$ for small $r$. Hence we can write the above integral in polar coordinates and obtain

$$f_k(x) = \int_0^x \int_{\theta_1(r)}^{\theta_2(r)} g_k(re^{i\theta})r d\theta dr,$$

where $\theta_1$ and $\theta_2$ vary smoothly with $r$. Since $g_k$ is infinitely differentiable and bounded on $V_k$, we see by repeated applications of the Leibniz integral rule [Fla73] that $f_k$ is also infinitely differentiable in $[0, \epsilon)$ for some $\epsilon > 0$ small enough.

**Definition 3.1.** Given digits $a_1, \ldots, a_n$, we define the cylinder set $\langle a_1, \ldots, a_n \rangle$ to be the set of all points in $K_d$ for which the nearest-integer $a_d$-continued fraction expansion is of the form $[0, a_1, \ldots, a_n, \ldots]$. We call the cylinder set admissible if it has an interior point.

We now state the the mixing property in [ENN23] that we require.

**Proposition 3.1** (Ei, Nakada, Natsui). Let $\mu_d$ be as in Lemma 3.1. There exist constants $\alpha > 0$ and $0 < \rho < 1$ depending on $d$ such that for all $k, m \in \mathbb{N}, b_1, b_2, \ldots, b_k \in \mathbf{a}_d$ and $c_1, \ldots, c_m \in \mathbf{a}_d$ we have

$$|\mu_d(\langle b_1, \ldots, b_k \rangle \cap G^{-(n+k)}(c_1, \ldots, c_m)) - \mu_d(\langle b_1, \ldots, b_k \rangle)\mu_d(\langle c_1, \ldots, c_m \rangle)| \leq \alpha \rho^k \mu_d(\langle b_1, \ldots, b_k \rangle)\mu_d(\langle c_1, \ldots, c_m \rangle).$$  

(11)

We define a piece of notation which will give a simple interpretation to this proposition. If $Q(a_1(z), a_2(z), \ldots)$ is a set of conditions concerning the digits of the continued fraction expansion, we write

$$\mathbb{P}(Q(a_1(z), a_2(z), \ldots)) = \mu_d\{z \in K_d : Q(a_1(z), a_2(z), \ldots) \text{ holds}\}.$$  

1Here the differentiability of a function at 0 means existence of the right derivative.
We may then rewrite $\mu_d(b_1, b_2, \ldots, b_k) \cap T^{-(n+k)}(c_1, \ldots, c_m)$ in (11) as
\[
P(a_1(z) = b_1, \ldots, a_k(z) = b_k, a_{k+n}(z) = c_1, \ldots, a_{k+n+m}(z) = c_m).
\]
Similarly, the quantities $\mu_d(b_1, \ldots, b_k)$ and $\mu_d(c_1, \ldots, c_m)$ can be thought of as $\mathbb{P}(a_1(z) = b_1, \ldots, a_k(z) = b_k)$ and $\mathbb{P}(a_1(z) = c_1, \ldots, a_m(z) = c_m)$ respectively. Equation (11) therefore tells us that finite sequences of digits behave almost independently with an error term that decreases exponentially with respect to the gap between the sequences. We will use the following technical proposition to show that this mixing condition is sufficient to obtain an extreme value theorem for the digits $|a_n|$.

**Proposition 3.2** (Leadbetter, Lindgren, Rootzén). Let $\xi_1, \xi_2, \ldots$ be a sequence of identically distributed but not necessarily independent random variables. For $u \in \mathbb{R} \cup \{\infty\}$ and natural numbers $i_1, \ldots, i_l \in \mathbb{N}$, let
\[
F_{i_1, \ldots, i_l}(u) = \mathbb{P}(\xi_{i_1} \leq u, \xi_{i_2} \leq \cdots, \xi_{i_l} \leq u).
\]
We assume the following almost-independence condition holds. If
\[
1 \leq i_1 < i_2 < \cdots < i_l < j_1 < \cdots < j_p,
\]
is an array of increasing integers, we have for all $u \in \mathbb{R} \cup \{\infty\}$ that
\[
|F_{i_1, \ldots, i_l} - F_{i_1, \ldots, i_l}(u)F_{j_1, \ldots, j_p}(u)| \leq \alpha(j_1 - i_l),
\]
with $\lim_{n \to \infty} \alpha(n) = 0$.
Assume furthermore that there is some sequence $(u_n)_n$ with $u_n \to \infty$ as $n \to \infty$ satisfying
\[
\lim_{n \to \infty} \sup_{n \geq [n/k]} \sum_{j=2}^{[n/k]} \mathbb{P}(\xi_1 > u_n, \xi_j > u_n) \to 0 \text{ as } k \to \infty. \tag{13}
\]
Then $\mathbb{P}(\max_{1 \leq k \leq n} \xi_k \leq u_n) \to e^{-\tau}$ as $n \to \infty$ for some $\tau$ if and only if $n(1 - F_1(u_n)) \to \tau$ as $n \to \infty$.

**Proof.** This is a weaker formulation of Theorem 3.4.1 in [LLR83].

We remark that (13) assures that the entries of $(\xi_k)_k$ which exceed $u_n$ are sufficiently sparse as $n \to \infty$. It is clear by Proposition (3.1) that (12) holds for $\xi_n = |a_n(z)|$. We now show that we can find a sequence $(u_n)_n$ such that (13) holds.

**Lemma 3.3.** Let $H$ be the constant in Lemma 3.2. If $y > 0$, then
\[
\limsup_{n \to \infty} \sum_{j=2}^{[n/k]} \mathbb{P}\left(|a_1(z)| > y \sqrt{\frac{n}{H}}, |a_j(z)| > y \sqrt{\frac{n}{H}}\right) \to 0 \text{ as } k \to \infty.
\]

**Proof.** By Lemma 3.1 and Lemma 3.2 we have that
\[
\mathbb{P}\left(\xi_1 > y \sqrt{\frac{n}{H}}, \xi_j > y \sqrt{\frac{n}{H}}\right) = \frac{1}{y^{n^2}} + O\left(\frac{\rho^j}{n^3}\right)
\]
for some $0 < \rho < 1$, as $j, n \to \infty$. We therefore see that the sum in the statement of the lemma is of order $O(1/k)$. □
Proof of Theorem 1.1 for \( \eta = \mu_d \). We show that the theorem holds for the measure \( \mu_d \) when \( C_d = H^{-1/2} \), where \( H \) is the constant in Lemma 3.2. By Lemma 3.3, we can apply Proposition 3.2 which states that it suffices to show that

\[
nP \left( |a_n(z)| > y \sqrt{\frac{n}{H}} \right) \to \frac{1}{y^2} \text{ as } n \to \infty.
\]

This follows immediately from Lemma 3.2. Extending this to any measure equivalent to the Lebesgue measure is a straightforward application of a result due to Eagleson, which we state below.

Let \((R_n)_n\) be a sequence of random variables on some probability space \((X, A, P)\) and let \(R\) be another random variable, not necessarily defined on \(X\). We write \(R_n \Rightarrow R\) if \(R_n\) converges to \(R\) in distribution, i.e., \(P \circ R_n^- \to \text{law of } R\) as \(n \to \infty\). We use the following theorem due to Eagleson, see \([Eag76, Zwe07]\).

**Proposition 3.3 (Eagleson).** Let \((R_n)_n\) be a sequence of random variables on some probability space \((X, A, P)\). Let \(R\) be another random variable and assume \(R_n \Rightarrow R\). Assume there exists a \(\sigma\)-finite measure \(\mu\) such that \(P\) is absolutely continuous with respect to \(\mu\), and an ergodic nonsingular map \(G\) on \((X, A, \mu)\) for which

\[
|R_n \circ G - R_n| \mu \Rightarrow 0.
\]

Then for all measures \(\eta\) which are absolutely continuous with respect to \(\mu\), we have that

\[
R_n \eta \Rightarrow R.
\]

Proof of Theorem 1.1 for \(\eta\) equivalent to the Lebesgue measure. Let \(R\) be some \((0, \infty)\)-valued random variable with cumulative distribution function \(F_R(y) = e^{-1/y^2}\). Theorem 1.1 for the measure \(\mu_d\) therefore states that

\[
R_N := \max_{n \leq N} |a_n(\beta)| \mu_d \Rightarrow R \text{ as } N \to \infty.
\]

Since \(a_n(G(\beta)) = a_{n+1}(\beta)\) for all \(n \in \mathbb{N}\), all \(y \geq 0\) satisfy

\[
\mu_d \{ \beta : |R_n \circ G - R_n| (\beta) \leq y \} \leq \mu_d \left\{ \beta : \frac{\max \{|a_1(\beta)|, |a_{N+1}(\beta)|\}}{C_d \sqrt{N}} \leq y \right\}.
\]

By Proposition 3.1, this tends to zero as \(N \to \infty\) if \(y > 0\). This proves \(|R_n \circ G - R_n| \mu \Rightarrow 0\), which means that we can apply Proposition 3.3 and obtain that

\[
R_N \eta \Rightarrow R,
\]

for any \(\eta\) which is absolutely continuous with respect to \(\mu_d\), or equivalently, the Lebesgue measure.
4 Bianchi Orbifolds

This section is dedicated to elucidating the geometry of the five Bianchi orbifolds under consideration. Let us start with a short description of the construction of the three dimensional hyperbolic space. We refer to [EGM98] for more information. Consider the subset of the quaternions defined by

\[ H^3 = \{ x + yi + rj : x, y \in \mathbb{R}, r > 0 \}. \]

With the Riemannian metric

\[ ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}, \]

it is the unique simply connected three dimensional Riemannian manifold of constant negative sectional curvature \(-1\). Let \( I \) be the \( 2 \times 2 \) identity matrix. For \( w \in H^3 \), there is a unique \( z \in \mathbb{C} \) and \( r > 0 \) such that \( w = z + rj \). In these coordinates, we can write the action of \( \text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\{\pm I\} \) on \( H^3 \) as

\[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\cdot
\begin{pmatrix}
    z + rj
\end{pmatrix}
= \frac{(az + b)(\overline{cz} + \overline{d}) + \alpha r^2}{|cz + d|^2 + |c|^2 r^2} + \frac{rj}{|cz + d|^2 + |c|^2 r^2}.
\]

(16)

The boundary \( \partial H^3 \) of \( H^3 \) is the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). The action \( (16) \) extends to the familiar action on \( \partial H^3 \) by Möbius transformations by setting \( r = 0 \) in \( (16) \).

4.1 Fundamental domains for Bianchi Groups

The following proposition lists a complete set of generators for the Bianchi groups under consideration. This result can be found in [Coh68].

**Proposition 4.1.** Let \( \omega = \frac{-1 + \sqrt{-d}}{2} \) if \( d = 3 \mod 4 \), and let \( \omega = \sqrt{-d} \) otherwise. Let \( \overline{\omega} \) be the complex conjugate of \( \omega \). Define

\[
S = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\text{ and } T^q = \begin{pmatrix}
1 & q \\
0 & 1
\end{pmatrix}
\]

for \( q \in \mathfrak{o}_d \). If \( d = 2, 7, \) or \( 11 \), then

\[ S, T^1 \text{ and } T^\omega \]

generate \( \text{PSL}_2(\mathfrak{o}_d) \). If \( d = 1, 3 \), then we must add

\[
\begin{pmatrix}
\omega & 0 \\
0 & \overline{\omega}
\end{pmatrix}
\]

to \( (17) \) to form a complete set of generators.

Geometrically, the map \( S \) is an inversion with respect to the unit hemisphere \( \{ x + yi + rj \in \mathbb{H}^2 : x^2 + y^2 + r^2 = 1 \} \) followed by a reflection across the \( yr \)-plane. The map \( T^q \) is the translation \( z + rj \mapsto z + q + rj \) parallel to the complex plane.

The five Bianchi groups we consider have a fundamental domain that is simple to describe.
Figure 2: Construction of the fundamental domain for $d = 2$. The points $x + yi + rj \in \mathcal{F}_2$ belong to the region bounded by the opaque hemisphere and the four vertical rectangles.

**Proposition 4.2.** If $d = 2, 7$ or $11$, a fundamental domain for $PSL_2(\mathbb{Q}_d)$ is given by

$$\mathcal{F}_d = \{z + rj \in \mathbb{H}^3 : z \in K_d, |z|^2 + r^2 > 1\}.$$  

For $d = 1, 3$, the above domain must be modified to account for the extra generator in Proposition 4.1. Define

$$K_1' = \left\{ x + iy : x \in \mathbb{C} : 0 < x < \frac{1}{2}, |y| < \frac{1}{2} \right\}.$$ 

and

$$K_3' = \left\{ z \in K_3 : \arg z \in \left[ \frac{\pi}{6}, \frac{5\pi}{6} \right] \right\}.$$ 

The fundamental domain is given by

$$\mathcal{F}_d = \{z + rj \in \mathbb{H}^3 : z \in K_d', |z|^2 + r^2 > 1\}$$  

for $d = 1, 3$.

**Proof of Proposition 4.2.** These results date back to Bianchi [Bia92]. An exposition can be found in [Swa71], Chapters 5, 6, 10, 13.

The orbifold $\mathbb{P}_d$ is noncompact and unbounded with respect to the hyperbolic metric. The unbounded end of $\mathbb{P}_d$ corresponding to $r \to \infty$ in $\mathcal{F}_d$ is called the **cusp**.
In what follows, when we say something holds almost everywhere, we mean with respect to either the Liouville measure on the relevant unit tangent bundle, or the Lebesgue measure on $K_d$.

## 5 From a Fréchet law to Cusp Excursions

In [Pol09] the geodesic flow on $M$ is modeled as a suspension flow over the natural extension of the real-valued Gauss map. As far as we are aware, there is no such known interpretation for nearest-integer continued fractions. Nicolas Chevallier obtains such a result for a different continued fraction expansion in [Nic21]. Lukyanenko and Vandehey [LV23] establish a partial result by considering so-called geodesic marking for various continued fraction expansions associated to a large class of hyperbolic orbifolds. Nevertheless, it is still possible to establish a correspondence between the digits of the continued fraction expansion and the height of these cusp excursions.

Let us first establish some geometrical intuition. In what follows let $H(b) \subset \mathbb{H}^3$ be the hemisphere of Euclidean radius one with centre $b \in \mathbb{C}$. Suppose we have a geodesic $\gamma$ on $T^1P_d$. Let $\gamma_0$ be a lift of this geodesic to $\mathbb{H}^3$. We say that $\gamma_0$ has endpoints $(z_1, z_2)$ if $z_1, z_2 \in \partial \mathbb{H}^3$ are endpoints of $\gamma_0$ and $z_1$ is the attracting endpoint. Let us assume that $\gamma_0$ satisfies $z_1 \in K_d$ and $|z_2| > 1$.

Let $z_1 = [0, a_1, a_2, \ldots]$ and let us first assume for simplicity that this expansion contains no small digits, i.e. $|a_n| > 3$ for all $n$. This geodesic intersects $H(0)$ in $\gamma_0(t_0)$ at some time $t_0$. By applying the map $T^{a_1}S$, we obtain another lift $\tilde{\gamma}_1$ of $\gamma$ with endpoints

$$(z_1^{(1)}, z_2^{(1)}) = -\left(\frac{1}{z_1} - a_1, \frac{1}{z_2} - a_1\right).$$

These endpoints once again satisfy the condition $z_1^{(1)} \in K_d$ and $\left|z_2^{(1)}\right| > 1$. Define $t_1$ to be the time at which $\tilde{\gamma}_1$ intersects $H(0)$. The map $S$ maps the interior of the unit hemisphere $H(0)$ to its exterior and vice versa. Hence the map $T^{a_1}S$ maps the exterior of $H(0)$ to the interior of $H(a_1)$ and maps $\tilde{\gamma}_0(t_0) \in H(0)$ to $\tilde{\gamma}_1(t_0) \in H(a_1)$. The time interval $[t_0, t_1]$ therefore corresponds to the time $\tilde{\gamma}_1$ spends above the two hemispheres. Take $n = 1$ in Figure 3 for an illustration. By symmetry of the choice function $\lfloor \cdot \rfloor_d$, we have that $z_1^{(1)} = [0, -a_2, -a_3, \ldots]$ for almost all values of $z_1$. We can therefore apply the map $T^{a_2}S$ to obtain a new geodesic lift $\tilde{\gamma}_2$. Iterating this process, we apply the maps $T^{a_1}S, T^{a_2}S, \ldots$ in order to obtain a sequence of lifts $\{\tilde{\gamma}_n\}_n$ of $\gamma$. Define the sequence $\{t_n\}_n$ by the relation $\tilde{\gamma}_n(t_n) \in H(0)$. Each interval $[t_{n-1}, t_n]$ corresponds to the segment of $\tilde{\gamma}_n$ above the hemispheres $H(0)$ and $H((-1)^{n-1}a_n)$. We refer once again to Figure 3.

To summarise: the segment $\gamma([t_{n-1}, t_n])$ of the geodesic $\gamma$ has a lift $\tilde{\gamma}_n([t_{n-1}, t_n])$ which corresponds to the segment of $\tilde{\gamma}_n$ above the hemispheres $H(0)$ and $H((-1)^{n-1}a_n)$. By definition of the metric

\[ \text{It is in fact always possible to find such a lift} \text{ [Abr22].} \]
Figure 3: The geodesic $\tilde{\gamma}_n$ on $H^3$ which arcs from the point $z_1^{(n)}$ to $z_2^{(n)}$. It intersects the spheres $H(a_n)$ and $H(0)$ at time $t_{n-1}$ and $t_n$ respectively.

The distance between this apex and the horocycle $\{z + rj : r = 1\}$ is the logarithm of the Euclidean height of the apex. Understanding the long term behavior of $\gamma$ therefore boils down understanding the asymptotic behavior of $t_n$ and the height of the apexes.

The assumption that there are no small digits is too restrictive, however. Indeed, a straightforward consequence of Proposition 3.1 is that the nearest-integer $\alpha_d$-continued fraction expansions of almost all complex numbers contain small digits. We discuss these technicalities in the next subsection.

5.1 Excursion Times

One complication that may arise with small digits is that the repelling endpoints do not necessarily satisfy the condition $|z_2^{(n)}| > 1$ and hence the times $t_n$ discussed in the previous subsection are not always defined. To fix this, we restrict to geodesics $\gamma$ on $T^1 P_d$ with a lift $\tilde{\gamma}_0$ where the repelling endpoint is $\infty$. If such a lift exists, we may assume its endpoints are $(\beta, \infty)$ for some $\beta \in K_d$. In the final proof of Theorem 1.2, we argue that for sufficiently large times $t$, generic geodesics exhibit the same statistical properties as these model geodesics.

We can also reparametrise $\gamma, \tilde{\gamma}_0$ so that $\tilde{\gamma}_0(0) = \beta + j$. As before, we define

$$\tilde{\gamma}_n := (T^{(-1)^{n-1}a_n} S) \circ \tilde{\gamma}_{n-1}$$

for all $n \in \mathbb{N}$. For almost all values of $\beta$, the endpoints of $\tilde{\gamma}_n$ are given
by 
\((-1)^n \hat{G}^n(\beta, \infty),\)
where \(\hat{G}\) is the extended Gauss map defined by
\[
\hat{G}(z, w) = \left( \frac{1}{z} - \left[ \frac{1}{z} \right], \frac{1}{w} - \left[ \frac{1}{z} \right] \right)
\]
for all \(z, w \in \hat{C}\) with \(z \neq w\). If \(z = [0, a_1, a_2, \ldots]\) and \(w = \infty\), we see that
\[
\hat{G}^n(\beta, \infty) = \begin{pmatrix} G^n(\beta), -a_n + \frac{1}{-a_n-1 + \frac{1}{\cdots -a_2 + \frac{1}{-a_1}}} \end{pmatrix} = \left( G^n(\beta), \frac{-q_n}{q_{n-1}} \right),
\]  \((18)\)
where the last equality is a consequence of (8). We also establish this result in the appendix in the proof of Lemma 5.1. Since \(\frac{q_n}{q_{n-1}} > 1\) the following is well-defined.

**Definition 5.1.** Let \(\beta = [0, a_1, a_2, \ldots] \in K_d\) and let \(\tilde{\gamma}_n\) be defined as above. We define the **intersection times** \(\{t_n\}_n\) by setting \(t_n\) to be the time the geodesic \(\tilde{\gamma}_n\) intersects with the unit hemisphere \(H(0) = \{ z + rj \in H^3 : |z|^2 + r^2 = 1 \}\) centred at the origin.

A problem may occur with small digits, however. Indeed, if \(|a_n| \leq 2\), then \(H(0)\) and \(H((-1)^{n-1}a_n)\) overlap. In particular, it is possible \(\tilde{\gamma}_n\) intersects the unit hemisphere \(H(0)\) first, which would imply that that \(t_{n-1} > t_n\). Nonetheless, we obtain the following result.

**Lemma 5.1.** Let \((p_n/q_n)_n\) be the convergents of \(\beta\) with respect to the nearest-integer \(o_d\)-continued fraction expansions. For all \(n \geq 1\),
\[
|t_n - 2 \log |q_n| - \frac{3}{2} \log \left( 1 - \frac{|q_{n-1}|^2}{q_n} \right) | \leq D,
\]  \((19)\)
for some constant \(D\) which depends only on \(d\).

The proof involves finding the matrix \(g \in \text{PSL}_2(o_d)\) such that \(\tilde{\gamma}_n = g\tilde{\gamma}_0\) and representing it in terms of the associated quotient pair of sequences. By a straightforward but somewhat lengthy calculation, we find an explicit formula for \(t_n\) from which the lemma follows. The details can be found in the appendix. The following definition provides a monotone sequence based on the intersection times.

**Definition 5.2.** Define the **excursion times** \(\{t^*_n\}_n\) by setting
\[
t^*_n = \max_{i \in \{1, 2, \ldots, n\}} t_i.
\]

Using a result in [ENN22], we find a subsequence of linear growth for which the third term in (19) is bounded. We also use the ergodic properties of the Gauss map to establish a growth rate for \((|q_n|)_n\) to obtain the following proposition.
Proposition 5.1. There exists a constant $C^* > 0$ such that for almost every $\beta \in K_d$, the excursion times $(t^*_n)_n$ in Definition 5.2 satisfy

$$\lim_{n \to \infty} \frac{t^*_n}{n} = C^*. $$

The details may be found in the appendix.

We establish a correspondence between the digits of the continued fraction expansion and the maximum height of the corresponding cusp excursion in the following lemma.

Lemma 5.2. Let $\tilde{w}_\beta$ be the tangent vector to $\tilde{\gamma}_0$ at $\tilde{\gamma}_0(0)$. Let $\tilde{w}_\beta(t)$ be the vector obtained by applying the geodesic flow for some time $t$. Let $w_\beta$ and $w_\beta(t)$ be their respective projections to $T_1\mathbb{P}_d$. The sequence

$$n \mapsto \max_{t \in [0, t^*_n + 1]} e^{d(w_\beta, w_\beta(t))} - \frac{1}{2} \max_{k \leq n} |a_k|$$

is bounded from above for almost all $\beta$.

By [18] we have that the Euclidean height of the apex of $\tilde{\gamma}_n$ is approximately equal to $|a_n|/2$. The hyperbolic distance to the horocycle $\{z + j : z \in \mathbb{C}\}$ is the logarithm of that. If $|a_n| > 2$, then $\tilde{\gamma}_n$ reaches this apex in the time interval $[t_{n-1}, t_n]$. The remainder of the argument involves proving the intuitive assumption that sequences of small digits do not correspond to cusp excursions. A detailed proof can be found in the appendix.

5.2 Proof of Theorem 1.2

Theorem 1.2 is a theorem about the Liouville measure $m$ on $T^1\mathbb{P}_d$. We define a surjective map from $T^1\mathbb{P}_d$ to $K_d$ for which the pushforward of $m$ is absolutely continuous with respect to the Lebesgue measure. This will allow us to apply Theorem 1.1. Recall that we defined $F_d$ to be the fundamental domain of the action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{H}^3$. For almost all $v$, there is a unique lift $v' \in T^1F_d$. Almost surely, there exists a unique $q \in \mathbb{Z}_d$ such that if we define

$$\tilde{v} := T^q v' = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} v',$$

then the oriented unit speed geodesic $t \mapsto \tilde{v}(t)$ with tangent vector $\tilde{v}$ at time $t = 0$ has attracting endpoint $\tilde{v}_\infty \in K_d$. Let $\pi : T^1\mathbb{P}_d \to K'_d$ be the map which sends $v \mapsto \tilde{v}_\infty$. The following result is easily seen.

Lemma 5.3. Let $m$ be the Liouville measure on $T^1\mathbb{P}_d$. The pushforward measure $m \circ \pi^{-1}$ on $K_d$ is absolutely continuous with respect to the Lebesgue measure.

Proof of Theorem 1.2 We first claim that elements in the same fibre of $\pi$ have the same asymptotic behaviour for large $T$. Let $v \in \pi^{-1}(\beta)$ and let $\tilde{w}_\beta, w_\beta$ be as in Lemma 5.2. By definition of $\pi$, there is a lift
The terms involving $t$ into (20) completes the proof. Elementary hyperbolic geometry shows that there exists a $\Delta \in \mathbb{R}$ such that $d(\tilde{w}_\beta(t + \Delta), \tilde{v}(t)) \to 0$ as $t \to \infty$. Hence $d(w_\beta(t + \Delta), v(t)) \to 0$ as $t \to \infty$, which proves the claim. Every $\alpha \in \mathbb{R}$ therefore satisfies

$$
\lim_{T \to \infty} \nu \left\{ v \in T^1 \mathbb{M} : \sup_{0 \leq t \leq T} d(v, v(t)) \leq \frac{1}{2} \log T \leq \log(y) + \alpha \right\}
$$

$$
= \lim_{T \to \infty} (\nu \circ \pi^{-1}) \left\{ \beta \in K_d : \sup_{0 \leq t \leq T} d(w_\beta, w_\beta(t)) \leq \frac{1}{2} \log T \leq \log(y) + \alpha \right\}
$$

$$
= \lim_{T \to \infty} (\nu \circ \pi^{-1}) \left\{ \beta \in K_d : \sup_{0 \leq t \leq T} e^{d(w_\beta, w_\beta(t))} \leq e^\alpha \sqrt{T} \right\}.
$$

Let $(t^*_n)_n$ as in Definition 5.2. Proposition 5.1 states that the number of cusp excursions $N$ after some time $T$ is typically approximatively equal to $T/C^*$. Lemma 5.2 shows that the maximum value $e^{d(w_\beta, w_\beta(t))}$ attained in these $N$ excursions is asymptotically equal to $\frac{1}{2} \max_{1 \leq n \leq N} |a_n(\beta)|$, so we obtain that the expression above is equal to

$$
\lim_{N \to \infty} (\nu \circ \pi^{-1}) \left\{ \beta \in K_d : \max_{1 \leq n \leq N} |a_n(\beta)| \leq 2e^\alpha \sqrt{CdNy} \right\}.
$$

If $\alpha$ is chosen such that $2e^\alpha \sqrt{C^*} = C_d$, where $C_d$ is the constant appearing in Theorem 1.1, then we obtain this theorem that the limit evaluates to $e^{-1/y^2}$, which is what we had to prove. \qed

6 Appendix

We first give an explicit formula for the intersection of a geodesic with the unit hemisphere. In what follows, let $H(b) \subset \mathbb{H}^3$ be the hemisphere of radius one with centre $b \in \mathbb{C}$.

**Lemma 6.1.** Let $\tilde{\eta}$ be a geodesic with endpoints $(\beta, \alpha)$ satisfying $|\beta| < 1$ and $|\alpha| > 1$. The intersection point $z + rj$ of $\tilde{\eta}$ and $H(0)$ satisfies

$$
r = \frac{1 - |\beta|^2}{|\alpha|^2 - |\beta|^2} \alpha - \beta \quad \text{and} \quad z = \beta + \frac{1 - |\beta|^2}{|\alpha|^2 - |\beta|^2} (\alpha - \beta).
$$

**Proof.** The geodesic $\tilde{\eta}$ is the half-circle from $\beta$ to $\alpha$ perpendicular to the complex plane. This can be parametrised by

$$[0, 1] \to H : t \mapsto z(t) + r(t)j, \quad \text{where} \quad z(t) = \beta + t(\alpha - \beta) \quad \text{and} \quad r(t) = \sqrt{t(1-t)|\alpha - \beta|}.
$$

(20)

This will intersect $H(0)$ if and only there exists a $t^* \in [0, 1]$ satisfying $|z(t^*)|^2 + |r(t^*)|^2 = 1$. Since $|z(t^*)|^2 = z(t^*)z(t^*)$, we obtain using (20) that this relation can be written as

$$
|\beta|^2 + t^2|\alpha - \beta|^2 + 2t^* \Re(\alpha - \beta \overline{\beta}) + t^*(1 - t^*)|\alpha - \beta|^2 = 1.
$$

The terms involving $t^2$ cancel. Solving for $t^*$ and substituting this into (20) completes the proof. \qed

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The following lemma is about the Möbius transformations used to generate \( \tilde{\gamma}_n \) in Section 5.

**Lemma 6.2.** Let \( \beta = [0, a_1, a_2, \ldots] \). For \( n \in \mathbb{N} \), let

\[
P(n, \beta) = (T^{(-1)^n-1}a_n S) \cdots (T^{-a_2}S)(T^a_1 S) \in PSL_2(o_d).
\]

Then

\[
P(n, \beta) = \left( \frac{q_n}{(-1)^{n-1}q_{n-1}} \frac{-p_n}{(-1)^n p_{n-1}} \right) \cdot (\pm I),
\]

where \( p_n/q_n = [0; a_1; \cdots; a_n] \) is the \( n \)-th convergent of \( \beta \), and we recall that each element of \( PSL_2(\mathbb{R}) \) has two representatives in \( SL_2(\mathbb{R}) \) which are additive opposites.

**Proof.** Note that \( T^a \pm S = \begin{pmatrix} \mp a & 1 \\ -1 & 0 \end{pmatrix} \). In particular it is clear the lemma holds for \( P(1, \beta) \). The general case follows from induction using the recursive formula for the associated quotient pair of sequences (3). Given two sequences \((a_n)_n\) and \((b_n)_n\) of positive numbers, we write \( a_n \preceq b_n \) if \( a_n = O(b_n) \) and \( b_n = O(a_n) \) as \( n \to \infty \). We establish a bounded equivalence between the intersection times and an expression involving associated quotient pair of sequences.

**Proof of Lemma 5.1.** Let \( n \in \mathbb{N} \). By symmetry of the Gauss map under reflection, we have for almost all \( \beta \) that

\[
(\begin{pmatrix} -1 \\ 0 \end{pmatrix})^{G_n(\beta, \infty)} = (P(n, \beta) \cdot \beta, P(n, \beta) \cdot \infty). \tag{21}
\]

By Lemma 6.2 we have that

\[
(-1)^n \hat{G}^n(\beta, \infty) = (-1)^n \left( G^n(\beta), -\frac{q_n}{q_{n-1}} \right).
\]

By Lemma 6.1 we have that the geodesic \( \hat{\gamma}_n \) intersects the hemisphere \( H(0) \) at some point \( z_n + r_n \hat{J} \) with

\[
r_n = \frac{G^n(\beta) + \frac{q_n}{q_{n-1}}}{\frac{q_n}{q_{n-1}}^2 - |G^n(\beta)|^2} \sqrt{(1 - |G^n(\beta)|^2) \left( \frac{\left| \frac{q_n}{q_{n-1}} \right|^2 - 1}{1 - |G^n(\beta)|^2} \right)} \quad \text{and}
\]

\[
z_n = (-1)^n \left( G^n(\beta) - \frac{1 - |G^n(\beta)|^2}{\frac{q_n}{q_{n-1}}^2 - |G^n(\beta)|^2} \left( G^n(\beta) + \frac{q_n}{q_{n-1}} \right) \right). \tag{22}
\]

This is equivalent to the geodesic \( \hat{\gamma}_0 = P(n, \beta)^{-1} \hat{\gamma}_n \) intersecting the hemisphere \( P(n, \beta)^{-1}H(0) \) at some point

\[
z' + r' \hat{J} := P(n, \beta)^{-1} \cdot (z_n + r_n \hat{J}).
\]
Since this is a point on $\tilde{\gamma}_0$, $z' = \beta$. By Lemma 6.2 we see that $P(n, \beta)^{-1} = \begin{pmatrix} (1)^n p_{n-1} & p_n \\ (1)^n q_{n-1} & q_n \end{pmatrix}$, so by (16) we have that

$$r' = \frac{1}{r_n^{-1} |(1)^n q_{n-1} z_n + q_n| |q_n|^{-1} |(1)^n q_{n-1} z_n + q_n|^2} \quad (23)$$

Some rearranging of terms shows that $|(1)^n q_{n-1} z_n + q_n|$ evaluates to

$$|(q_{n-1}) \left( G^n(\beta) + \frac{q_n}{q_{n-1}} \right) - \frac{1}{\left| \frac{q_n}{q_{n-1}} \right|^2 - |G^n(\beta)|^2} \left( G^n(\beta) + \frac{q_n}{q_{n-1}} \right) |$$

$$= |q_{n-1}| \left( \frac{q_n}{q_{n-1}} \right)^2 - 1 \left( \frac{q_n}{q_{n-1}} \right)^2 - \frac{|G^n(\beta)|^2}{\left| \frac{q_n}{q_{n-1}} \right|^2} \left( G^n(\beta) + \frac{q_n}{q_{n-1}} \right) |$$

Since $G^n(\beta) \in K_d$ and $\sup_{z \in K_d} |z| < 1$, we obtain from the above equation and (22) that

$$|(1)^n q_{n-1} z_n + q_n| \asymp |q_{n-1}| \left( \left| \frac{q_n}{q_{n-1}} \right|^2 - 1 \right) \left| \frac{q_{n-1}}{q_n} \right|$$

and

$$r_n \asymp \left| \frac{q_{n-1}}{q_n} \right| \left( \left| \frac{q_n}{q_{n-1}} \right|^2 - 1 \right)^{1/2} \quad (24)$$

Since $r_n < 1$, we obtain by (23) that $1/r' \asymp r_n^{-1} |(1)^n q_{n-1} z_n + q_n|^2$. By substituting the bounded equivalencies in (24) we see that

$$e^{\tau_n} = \frac{1}{r'} \asymp \left| \frac{q_{n-1}}{q_n} \right|^3 \left( \left| \frac{q_n}{q_{n-1}} \right|^2 - 1 \right)^{3/2}$$

as $n \to \infty$. The lemma follows from the identity $\frac{|q_{n-1}|^3}{|q_n|^3} = |q_n|^2 \left( \left| \frac{q_n}{q_{n-1}} \right|^2 \right)^{3/2} \quad \square$

The following lemma is a fast growth property of the denominator of continued fraction expansions.

**Lemma 6.3.** For each $d \in \{1, 2, 3, 5, 7, 11\}$, there exists a number $r_d > 1$ and $M_d \in \mathbb{N}$ such that for each $\beta \in K_d \setminus \mathbb{Q}$, there exists an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ satisfying

$$1 \leq n_{k+1} - n_k \leq M_d \quad \text{and} \quad \left| \frac{q_{n_k}}{q_{n_k-1}} \right| \geq r_d$$

for all $k \in \mathbb{N}$.

**Proof.** We estimate the numerator $q_n$ of the continued fraction expansion in terms of $|q_n \beta - p_n|$. We first note the identity

$$|q_n \beta - p_n|^{-1} = |\beta| |G(\beta)||G^2(\beta)| \cdots |G^n(\beta)| \quad (25)$$

as $n \to \infty$. The lemma follows from the identity $\frac{|q_{n-1}|^3}{|q_n|^3} = |q_n|^2 \left( \left| \frac{q_n}{q_{n-1}} \right|^2 \right)^{3/2} \quad \square$
holds. This can be proved by induction. It can also be proved by considering the transformation \( P(n + 1, \beta) \) defined in Lemma 6.2 and evaluating the derivative of \( z \mapsto P(n + 1, \beta) \cdot z \) at \( z = \beta \) directly and with the chain rule. Proposition 5 in [ENN22] uses an argument involving Ford spheres to show that for all \( n > 0 \) and all irrational \( \beta \in K_d \),

\[
|q_n| |\beta q_n - p_n| = \frac{1}{|G^{n+1}(\beta) + \frac{q_{n+1}}{q_n}|}.
\]  

(26)

If \( A_d = \sup \{ z \in K_d |z| \} \), we see by the triangle inequality and the reverse triangle inequality that

\[
\frac{1}{2} |\beta q_n - p_n| \leq |q_{n+1}| \leq \frac{1}{(1 - A_d)} |\beta q_n - p_n|.
\]  

(27)

From the above and (25), we obtain for all \( m, n \in \mathbb{N} \) that

\[
\frac{|q_{n+m}|}{q_n} \geq \frac{1 - A_d}{2} \frac{1}{|G^{n+1}(\beta)| \cdots |G^{n+m}(\beta)|} \geq \frac{1 - A_d}{2} A_d^{-m}.
\]  

(28)

Let \( 1 < r_d < A_d^{-1} \). If the lemma were false, there would be arbitrarily large intervals \( \{n, n + 1, \ldots, n + m\} \) on which some \( \beta \in K_d \) satisfies \(|q_{n+1}/q_{n+1}| \leq r_d \) for \( 0 \leq i < m \) and hence \(|q_{n+m}/q_n| \leq r_d^n \), which contradicts (28) for \( m \) large enough.

\[\square\]

**Proof of Proposition 5.1** We first note that by taking the logarithm in (25) and using (28), we obtain the relation

\[
\lim_{n \to \infty} \frac{\log |q_n|}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log \left( |G^k(\beta)| \right),
\]

provided the latter limit exists. However, since the log function is integrable on \( K_d \) with respect to the lebesgue measure, it is integrable with respect to \( \mu_d \). By the Birkhoff ergodic theorem, almost all \( \beta \in K_d \) have denominators \((q_n)_n \) satisfying

\[
\lim_{n \to \infty} \frac{\log |q_n|}{n} = \int \log |z| d\mu_d(z) \in (0, \infty).
\]

It thus suffices to prove that

\[
\lim_{n \to \infty} \frac{t^*_n}{n} = 2 \lim_{n \to \infty} \frac{\log |q_n|}{n}.
\]

Lemma 6.3 shows that there exists a constant \( M_d \) and a sequence \((c(n))_n \in \mathbb{N}^d \) such that for all \( n \in \mathbb{N} \) we have \( 0 \geq n - c(n) \leq M_d \) and \(|q_{c(n)}/q_{c(n)-1}| \geq r_d \) for some constant \( r_d > 1 \). It follows from Lemma 5.1 that

\[
t_{c(n)} \asymp 2 \log |q_{c(n)}| \asymp t^*_{c(n)},
\]

where the latter bounded equivalence follows from the fact that \( t^*_{c(n)} = t_m \) for some \( m < c(n) \), with \( t_m \) bounded from above by \( D + 2 \log |q_m| \), with \( D \) the constant appearing in Lemma 5.1. Consequently, \( \lim_{n \to \infty} \frac{|q_{c(n)}|}{c(n)} = \lim_{n \to \infty} t^*_{c(n)}/c(n) \). However, both \( |q_n| \) and \( t^*_n \) are monotone sequences, so we may replace \( c(n) \) with \( n \) in this equality and obtain the proposition.

\[\square\]
Proof of Lemma 5.2. Fix \( r_d > 1 \) as in Lemma 6.3 and let \((n_k)_k\) be the sequence of all integers satisfying \(|q_{n_k}/q_{n_k-1}| \geq r_d\). Define the subsequence \((m_k)_k\) of \((n_k)_k\) by removing the entries \(n_k\) for which

\[
t_{n_k} \leq t_{n_k-1},
\]

i.e. \(m_k\) is the largest subsequence for which \(t_{m_k}\) is increasing. Note that by Lemma 5.1 we have that \(t_{n_k} \approx 2 \log |q_{n_k}|^2\) as \(k \to \infty\). Since \(|q_{n_k}|\) increases exponentially in \(k\), the number of consecutive entries that can be removed is bounded, and hence \(m_{k+1} - m_k\) is a bounded sequence as well. In fact, since the constant \(r_d > 1\) of Lemma 6.3 is a lower bound for the exponential growth rate, we can bound this sequence independently of the endpoint \(\beta\). Furthermore, there exists an \(M > 0\) such that if \(|a_n| > M\) for some \(n \in \mathbb{N}\), then \(|q_n/q_{n-1}| \geq r_d\) and \(t_n \geq t_{n-1}^*\). In other words, \(n \in \{m_k\}_k\) if \(|a_n| > M\).

We first study the behaviour of the geodesic \(\gamma\) defined in the introduction to Subsection 5.1 on the intervals \([t_{m_k-1}, t_{m_k}]\). To simplify the notation somewhat, let \(r = m_k - 1\) and \(s = m_k\). Take the sequence of lifted geodesics \(\tilde{\gamma}_n\) as in Subsection 5.1. We then have

\[
\tilde{\gamma}_s = T^{-(1)^{s-1}a_s} SQ(k) \tilde{\gamma}_r, \text{ where}
\]

\[
Q(k) = \left( S \prod_{l=r+1}^{s-1} T^{-(1)^{l-1}a_l} S \right) \tilde{\gamma}_r.
\]

Let \(w_1^*(k)\) be the intersection of \(\tilde{\gamma}_r\) with \(H(0)\) at time \(t_r\). On the interval \([t_r, t_s] = [t_{m_k-1}, t_{m_k}]\), the geodesic \(\tilde{\gamma}_s\) travels from

\[
w_1(k) := T^{-(1)^{s-1}a_s} Q(k) w_1^*(k)
\]

to its intersection \(w_2(k)\) with \(H(0)\). By Lemma (6.1) and the fact that \(|q_r/q_{r-1}|, |q_s/q_{s-1}| > r_d\), we have that the Euclidean heights of \(w_1^*(k)\) and \(w_2(k)\) are bounded positively from below for all \(k\). The implication (30) means only a finite number of digits can appear in the definition of \(Q(k)\). Boundedness of the sequence \(m_k - m_{k-1}\) therefore implies that the set \(\{Q(k) : k \in \mathbb{N}\}\) is finite. This shows that the Euclidean height of \(w_1(k)\) is bounded from below by a positive number over all values of \(k\).

By (21), the apex of \(\tilde{\gamma}_m\) is some point \(\tilde{v} := z_k + h_k j\) with \(z_k \in \mathbb{C}\) and \(|h_k - |a_s|/2| < 1\). Geometric considerations show that the hyperbolic
distance between \( w_\beta \) and the projection of \( \tilde{v} \) onto \( T^1P_d \) is given by 
\[
\log(h_k) + O(k^{-1}).
\]

If \( a_s \) is large enough, then \( \tilde{v} \) lies above both \( H(0) \) and \( T^{-(-1)^s}Q(k)H(0) \), and so the geodesic reaches \( \tilde{v} \) at some time in the interval \([t_r, t_s]\). Since \( w_1(k), w_2(k) \) are uniformly bounded from below for all \( k \), we have for sufficiently large \( |a_{m_k}| \) that the maximum distance to the projection of the horosphere \( \{z + r_j : r = 1\} \) on \( T^1P_d \) reached in the time interval occurs near the apex. Therefore,
\[
\max_{t \in [0, t_m]} e^{d(w_\beta, w_\beta(t))} - \frac{1}{2} \max_{l \leq n} |a_m|
\]
is a bounded sequence in \( k \), provided that there is a \( k \) for which \( |a_{m_k}| \) is large enough. By (30) Theorem 1.1, this is almost always guaranteed to happen. The implication (30) also shows that we may replace \( \max_{l \leq n} |a_m| \) above with \( \max_{l \leq m} |a_l| \). The lemma follows.

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