Brownian Motion on a Riemannian Manifold

Part III Essay for Master of Mathematics of the Mathematical Tripos, University of Cambridge

Karen Habermann

May 2013

Contents

Introduction

When particles are suspended in a fluid, they are found in a very animated and irregular state of motion. A description of such a phenomenon can already be found in an article by Jan Ingenhousz published in 1784. Nevertheless, nowadays this motion is known as Brownian motion, named after Robert Brown. He observed this phenomenon in 1827 while studying pollen grains of the plant *Clarckia pulchella*. The first persons who put Brownian motion in a mathematical framework were Thorvald Thiele in 1880 and Louis Bachelier in 1900. However, it was not until the work of Albert Einstein in 1905 and of Marian Smoluchowski in 1906 that the theory of Brownian motion really got started, cf. [14]. A nice collection of historical articles regarding Brownian motion, including Ingenhousz's and Brown's work, can be found under [7].

This essay serves as an introduction to the well-developed field of Brownian motion on a Riemannian manifold. The main two chapters are Chapters 4 and 5. After giving a definition of Brownian motion on a Riemannian manifold at the beginning of Chapter 4, we continue by discussing characterisations of Brownian motion using stochastic differential equations, in terms of discrete approximations and via the heat equation. In Chapter 5, we then analyse the recurrence and transience behaviour of Brownian motion. We conclude by presenting concrete examples of Riemannian manifolds for which one can decide whether Brownian motion on them is recurrent or transient. For the most part, these two chapters are based on Emery [3], Feres [4], Grigor'yan [5] and Hsu [9], with the details added in.

We shall assume basic knowledge of Differential Geometry, including manifolds and their tangent spaces. The definitions of any further concepts we need from Differential Geometry are given in Chapter 2. Moreover, in Chapter 3 we set up all the required notions from Stochastic Calculus on manifolds. However, we shall assume familiarity with probability theory up to Stochastic Calculus on $\mathbb{R}$. A reader who is unfamiliar with the definitions of filtrations, stopping times, real-valued semimartingales or the Itô integral could consult Rogers, Williams [15] and [16] beforehand.

In Chapter 1, we recall the definition of Brownian motion on $\mathbb{R}^d$. Furthermore, in order to motivate the different characterisations of manifold-valued Brownian motion, which we give in Chapter 4, we briefly present alternative ways of characterising Brownian motion on $\mathbb{R}^d$.

Throughout the entire essay we adopt the following conventions. Our manifolds are smooth and the underlying topological space is assumed to be Hausdorff, second-countable and connected. We write T_xM to denote the tangent space to the manifold M at the point $x \in M$ and $C^n(M)$ for the space of n-times continuously differentiable functions from M to $\mathbb{R}$. Moreover, we always work on a complete probability space and any filtration is assumed to satisfy the usual conditions, i.e. it is right-continuous and contains all the null sets. We also only consider continuous semimartingales which allows us to simply call them semimartingales. Finally, we use $[\cdot,\cdot]$ to denote the quadratic covariation of two semimartingales, whereas $\langle\cdot,\cdot\rangle$ is reserved for the Euclidean inner product.

Acknowledgment

I would like to thank Professor Norris for very helpful discussions and for setting such an interesting essay topic in the first place.

1 Motivation

To begin with, we recall the definition of a standard Brownian motion on $\mathbb{R}^d$.

Definition 1.1 *Let $X = (X_t)_{t\geq 0}$ be a continuous stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathbb{R}^d$. We call X a standard Brownian motion on $\mathbb{R}^d$ if it satisfies that*

(i) $X_0 = 0$,

(ii) *for any $0 \leq t_0 < t_1 < \ldots < t_n$, the increments $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_1} - X_{t_0}$ are independent, and*

(iii) *for any $0 \leq s < t$, the increment $X_t - X_s$ has normal distribution $\mathcal{N}(0, (t-s)I_d)$, where I_d denotes the identity matrix.*

We adopt the convention that any stochastic process $Y = (Y_t)_{t\geq 0}$ given by $Y_t = X_t + a$ for a standard Brownian motion $X = (X_t)_{t\geq 0}$ on $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ is called a Brownian motion on $\mathbb{R}^d$. Moreover, if $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ we say that X is a Brownian motion in $(\mathcal{F}_t)_{t\geq 0}$ if X is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and for all $s \geq 0$ the process $(X_{t+s} - X_s)_{t\geq 0}$ is independent of $\mathcal{F}_s$. In particular, $X = (X_t)_{t\geq 0}$ is then a martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Let $\mathbb{P}_a$ denote the law of a Brownian motion $X = (X_t)_{t\geq 0}$ on $\mathbb{R}^d$ starting at $a \in \mathbb{R}^d$. By property (iii) in Definition 1.1, it follows that for $t > 0$ and any Borel subset C of $\mathbb{R}^d$

$$\mathbb{P}_a(X_t \in C) = \frac{1}{(2\pi t)^{d/2}} \int_C \mathrm{e}^{-\|y-a\|^2/2t} \, \mathrm{d}y \, .$$

In Section 4.3, we establish a generalised version of this formula valid for Brownian motion on a Riemannian manifold.

The following theorem says that one can obtain a real-valued standard Brownian motion as the limit of a sequence of random walks on the integers scaled in the right way. Thereby, we use $\lfloor t \rfloor$ to denote the greatest integer less than or equal to t.

Theorem 1.2 (Donsker's invariance principle) *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed real-valued random variables with $\mathbb{E}[\xi_1] = 0$ and $\mathbb{E}[\xi_1^2] = \sigma^2 < \infty$. Let*

$$S_k = \sum_{i=1}^{k} \xi_i$$

denote the k^{th} partial sum. Then the stochastic processes $X^{(n)} = \left(X_t^{(n)}\right)_{t\geq 0}$ which are defined by

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} R_{nt} \, , \qquad \text{where} \qquad R_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1} \, ,$$

are continuous and converge weakly to a Brownian motion on $\mathbb{R}$.

A proof is given in Karatzas, Shreve [11, Chapter 2]. To approximate a standard Brownian motion on $\mathbb{R}^d$ by scaled random walks on $\mathbb{Z}^d$, one can take d independent sequences

$$\left(\xi_{1,n}\right)_{n\in\mathbb{N}}, \left(\xi_{2,n}\right)_{n\in\mathbb{N}}, \ldots, \left(\xi_{d,n}\right)_{n\in\mathbb{N}}$$

of independent and identically distributed real-valued random variables with zero mean and finite variance. If we define $X_i^{(n)}$ in terms of $\left(\xi_{i,m}\right)_{m\in\mathbb{N}}$ as $X^{(n)}$ is defined in terms of $\left(\xi_m\right)_{m\in\mathbb{N}}$ then

$$\left(X_1^{(n)}, X_2^{(n)}, \ldots, X_d^{(n)}\right)$$

converges weakly to a Brownian motion on $\mathbb{R}^d$ as $n \to \infty$.

In Proposition 1.5, we find another alternative characterisation of Brownian motion on $\mathbb{R}^d$. This new characterisation has the advantage that it can be generalised to a Riemannian manifold. The latter is done in Chapter 4.

For the rest of this chapter, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. A continuous stochastic process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ is called a semimartingale on $\mathbb{R}^d$ if and only if $X^i = (X^i_t)_{t\geq 0}$ is a semimartingale on $\mathbb{R}$ for each $1 \leq i \leq d$. Similarly, one defines $\mathbb{R}^d$-valued local martingales. Moreover, we have the following two important theorems. For proofs we refer to Rogers, Williams [16, VI. 39 and IV. 33].

Theorem 1.3 (Itô's formula) *If $X = (X_t)_{t\geq 0}$ is a semimartingale on $\mathbb{R}^d$ and $f \in C^2(\mathbb{R}^d)$ then $(f(X_t))_{t\geq 0}$ is a semimartingale on $\mathbb{R}$ satisfying*

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\,\mathrm{d}X^i_s + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\,\mathrm{d}\left[X^i, X^j\right]_s .$$

Theorem 1.4 (Lévy's characterisation of Brownian motion) *An $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ if and only if X is a local martingale and $\left[X^i, X^j\right]_t = t\delta^{ij}$ for $1 \leq i, j \leq d$.*

As shown in the proof of the next proposition, Lévy's characterisation is a nice tool to determine whether a given semimartingale is a Brownian motion. In the following, let Δ denote the usual Laplace operator on $\mathbb{R}^d$, i.e.

$$\Delta = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x^i}\right)^2 .$$

Proposition 1.5 *An $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ if and only if for all $f \in C^\infty(\mathbb{R}^d)$ the process $N = (N_t)_{t\geq 0}$ given by*

$$N_t = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta f(X_s)\,\mathrm{d}s$$

is a local martingale on $\mathbb{R}$.

Proof. Suppose X is a Brownian motion. From Lévy's characterisation, we know $\left[X^i, X^j\right]_t = t\delta^{ij}$ for $1 \leq i, j \leq d$. Thus, we have

$$\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\,\mathrm{d}\left[X^i, X^j\right]_s = \sum_{i=1}^{d} \frac{\partial^2 f}{(\partial x^i)^2}(X_s)\,\mathrm{d}s = \Delta f(X_s)\,\mathrm{d}s$$

and Itô's formula yields

$$N_t = \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\,\mathrm{d}X^i_s .$$

By Lévy's characterisation, we also know that X is a local martingale, i.e. X^i is a local martingale for each $1 \leq i \leq d$. Since the Itô integral preserves local martingales, it follows that N is indeed a local martingale.

Conversely, suppose that $f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta f(X_s)\,\mathrm{d}s$ is a local martingale for each $f \in C^\infty(\mathbb{R}^d)$. By taking $f = x^i$, which is a smooth function on $\mathbb{R}^d$ satisfying $\Delta f \equiv 0$, we deduce that

$$N^i_t = X^i_t - X^i_0$$

is a local martingale for each $1 \leq i \leq d$. In particular, $X = (X_t)_{t \geq 0}$ is a local martingale. Furthermore, by considering $f = x^i x^j$, which satisfies $\Delta f = 2\delta^{ij}$, we obtain that

$$\begin{aligned} N_t^{ij} &= X_t^i X_t^j - X_0^i X_0^j - \frac{1}{2} \int_0^t 2\delta^{ij} \, \mathrm{d}s \\ &= X_t^i X_t^j - X_0^i X_0^j - t\delta^{ij} \end{aligned} \tag{1.1}$$

is a local martingale. However, if N and $\widetilde{N}$ are local martingales on $\mathbb{R}$ then $\left[N, \widetilde{N}\right]$ is the unique continuous adapted process $A = (A_t)_{t \geq 0}$ of finite variation with $A_0 = 0$ and such that $(N_t \widetilde{N}_t - A_t)_{t \geq 0}$ is a local martingale. Hence, (1.1) implies that $\left[X^i, X^j\right]_t = t\delta^{ij}$. From Lévy's characterisation, we conclude that X is a Brownian motion on $\mathbb{R}^d$, as claimed. $\square$

In Chapter 4, we show that the preceding proposition is in fact a special case of Theorem 4.7, which is taken from Hsu [9]. The proof we gave for Proposition 1.5 is a simplified version of the proof of Theorem 4.7. However, it does still illustrate the main ideas used in the more general proof.

2 Concepts needed from Differential Geometry

In the next two chapters, we set up the Differential Geometry and the Stochastic Calculus needed to discuss Brownian motion on a Riemannian manifold.

Remark 2.1 *In this chapter only, we use X and Y to denote vector fields. Throughout the rest of the essay, the letters X and Y are reserved for stochastic processes.*

2.1 Riemannian manifolds and the Laplace-Beltrami operator

Working on $\mathbb{R}^d$ we always have the Euclidean inner product at hand, and for instance, we need it to define the usual Laplace operator Δ. When we aim to generalise Brownian motion to manifolds, we therefore want to work on manifolds which are equipped with a Riemannian metric.

Definition 2.2 *Let M be a manifold. A Riemannian metric on M is a smooth tensor field $h \in \Gamma(T^*M \otimes T^*M)$ such that $h_x \in T_x^*M \otimes T_x^*M$ is symmetric and positive definite for all $x \in M$.*

By our topological assumptions on a manifold M, it always admits a Riemannian metric.

Definition 2.3 *A Riemannian manifold is a pair (M, h) consisting of a manifold M and a Riemannian metric h on M.*

For a Riemannian manifold (M, h), we adopt the shorthand notation M with the presence of the Riemannian metric h being understood. We also set $d = \dim M$.

Our next aim is to define the Laplace-Beltrami operator Δ_M on a Riemannian manifold M. It is a well-known fact from Differential Geometry, e.g. see Kobayashi, Nomizu [12, Chapter 4], that for a given Riemannian manifold M there exists a unique covariant derivative ∇ on M which is metric, i.e.

$$X(h(Y,Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z) \quad \text{for all } X, Y, Z \in \Gamma(TM) \ ,$$

and torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0 \quad \text{for all } X, Y \in \Gamma(TM) \ .$$

The unique such covariant derivative is called the Levi-Civita connection.

Let $f \in C^\infty(M)$ be a smooth function on M and let $x \in M$ be fixed. Since h_x is positive definite, we can define $(\operatorname{grad} f)_x$ to be the unique vector $v_0 \in T_xM$ satisfying

$$h(v_0, v) = (\mathrm{d}f)_x(v) = v(f) \quad \text{for all } v \in T_xM \ .$$

This gives rise to a vector field $\operatorname{grad} f$ on M. In the following, let $(E_1, E_2, \dots, E_d)$ be a local orthonormal frame. For a vector field $X \in \Gamma(TM)$ we define the divergence $\operatorname{div} X$ of X by

$$\operatorname{div} X = \sum_{i=1}^{d} h(\nabla_{E_i} X, E_i) \ ,$$

where ∇ is the Levi-Civita connection on M. Since $\operatorname{div} X$ is the trace of $\nabla X \colon TM \to TM$ it does not depend on the choice of the local orthonormal frame $(E_1, E_2, \dots, E_d)$.

Since $\operatorname{grad} f$ is a vector field on M, one can particularly consider $\operatorname{div}\operatorname{grad} f$. It is clear that $\operatorname{div}\operatorname{grad} f$ is also well-defined for functions $f \in C^2(M)$. In fact, the Laplace-Beltrami operator Δ_M is applied to twice-continuously differentiable functions $f \in C^2(M)$ and yields a new function $\Delta_M f$ on M defined by

$$\Delta_M f = \operatorname{div}\operatorname{grad} f \ .$$

For instance, if we consider the manifold $M = \mathbb{R}^d$ equipped with the Euclidean inner product, then Δ_M agrees with the usual Laplace operator Δ on $\mathbb{R}^d$.

In our discussions in Chapter 4 we need the following lemma, which can be found in Hsu [9]. Thereby, $\nabla^2 f = \nabla(\mathrm{d}f)$ is the Hessian of a function $f \in C^2(M)$.

Lemma 2.4 *Let M be a Riemannian manifold. For $f \in C^2(M)$ it holds true that*

$$\Delta_M f = \sum_{i=1}^{d} \nabla^2 f(E_i, E_i) \; ,$$

i.e. $\Delta_M f$ equals the trace of $\nabla^2 f$.

Proof. From the definition of $\Delta_M f$, we have

$$\Delta_M f = \sum_{i=1}^{d} h(\nabla_{E_i} \operatorname{grad} f, E_i) \; .$$

By using the definition of $\operatorname{grad} f$ and the fact that ∇ is compatible with the metric, we deduce

$$\begin{aligned} h(\nabla_{E_i} \operatorname{grad} f, E_i) &= E_i(h(\operatorname{grad} f, E_i)) - h(\operatorname{grad} f, \nabla_{E_i} E_i) \\ &= E_i(\mathrm{d}f(E_i)) - \mathrm{d}f(\nabla_{E_i} E_i) \\ &= (\nabla_{E_i} \mathrm{d}f)(E_i) \\ &= \nabla^2 f(E_i, E_i) \end{aligned}$$

for each $1 \leq i \leq d$. Summing over i yields the desired expression. □

2.2 Principal bundles and connections

For one of the characterisations of Brownian motion on a Riemannian manifold, we need an understanding of principal bundles and connections defined on such bundles. Here, we only give the formal definitions, which one finds in many classical textbooks on Differential Geometry, e.g. in Kobayashi, Nomizu [12, Chapter 1 and 2].

To define a principal bundle, we need to know what a Lie group is.

Definition 2.5 *A Lie group G is both a manifold and an algebraic group with the additional property that the map*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, a) &\longmapsto g a^{-1} \end{aligned}$$

is smooth.

The general linear group $\mathrm{GL}(d)$ and the orthogonal group $\mathrm{O}(d)$ of degree d over the real numbers are examples of Lie groups. We meet both groups again when we present examples of principal bundles.

Definition 2.6 *Let M, P be manifolds, let G be a Lie group and let $\pi \colon P \to M$ be a smooth map. We call (P, π, M, G) a principal bundle over M with structure group G if the following three conditions are satisfied.*

(i) *G acts freely on P on the right, i.e. there is an action*

$$\begin{aligned} P \times G &\longrightarrow P \\ (p, g) &\longmapsto R_g(p) = p \cdot g \end{aligned}$$

with the property that if $R_g(p) = p$ for some $p \in P$ then g is the unit element of G.

(ii) *For $p_1, p_2 \in P$, there exists some $g \in G$ with $p_2 = R_g(p_1)$ if and only if $\pi(p_1) = \pi(p_2)$.*

(iii) *For any $x \in M$ there exists an open set $U_x \subset M$ and a diffeomorphism $\psi_x\colon \pi^{-1}(U_x) \to U_x \times G$ such that $\psi_x = (\pi, \phi_x)$ for some map $\phi_x\colon \pi^{-1}(U_x) \to G$ satisfying*

$$\phi_x(p \cdot g) = \phi_x(p)g$$

for all $p \in \pi^{-1}(U_x)$ and all $g \in G$.

One calls P the total space and M the base space of the principal bundle. Moreover, $P_x = \pi^{-1}(x)$ is called the fibre over x. By conditions (i) and (ii), the action is free and transitive on each fibre. Furthermore, the diffeomorphisms ψ_x are also called local trivialisations and π is called projection. We note that π must be surjective by condition (iii).

Our first example of a principal bundle is the so-called trivial bundle.

Example 2.7 *Let M be a manifold and let G be a Lie group. We claim that $(M \times G, \pi, M, G)$ is a principal bundle with $\pi\colon M \times G \to M$ being the projection onto the first factor and with the action of G on $M \times G$ given by*

$$R_{g_2}\left((m, g_1)\right) = (m, g_1 g_2)$$

for $(m, g_1) \in M \times G$ and $g_2 \in G$. Indeed, for any $x \in M$ we can choose

$$U_x = M \quad \text{and} \quad \psi_x = (\mathrm{id}, \mathrm{id})\colon M \times G \to M \times G$$

to satisfy condition (iii) *of Definition* 2.6*.*

In the following, we frequently denote a principal bundle (P, π, M, G) by P only. With that notation, the next example introduces the frame bundle $\mathscr{F}(M)$ over a manifold M.

Example 2.8 *Let M be a manifold of dimension d. For a fixed $x \in M$, let $\mathscr{F}(M)_x$ be the set of all linear isomorphisms $u\colon \mathbb{R}^d \to T_xM$. One generally calls an element $u \in \mathscr{F}(M)_x$ a frame at x. We now set*

$$\mathscr{F}(M) = \bigcup_{x \in M} \mathscr{F}(M)_x \ .$$

For each $u \in \mathscr{F}(M)$ there exists a unique $x \in M$ such that $u \in \mathscr{F}(M)_x$. Thus, we can define a map $\pi\colon \mathscr{F}(M) \to M$ by $\pi(u) = x$. Furthermore, an element $g \in \mathrm{GL}(d)$ can be considered as a linear isomorphism $g\colon \mathbb{R}^d \to \mathbb{R}^d$. Therefore, if $g \in \mathrm{GL}(d)$ acts on the right of $u \in \mathscr{F}(M)_x$ by composition, then

$$u \cdot g\colon \mathbb{R}^d \xrightarrow{g} \mathbb{R}^d \xrightarrow{u} T_xM$$

is a linear isomorphism, i.e. $u \cdot g \in \mathscr{F}(M)_x$. In particular, this action of $\mathrm{GL}(d)$ on $\mathscr{F}(M)$ preserves the fibres $\mathscr{F}(M)_x$ and is transitive on each fibre. Moreover, by using the charts on M one can give $\mathscr{F}(M)$ the structure of a manifold such that condition (iii) *in Definition* 2.6 *is also satisfied. This makes $(\mathscr{F}(M), \pi, M, \mathrm{GL}(d))$ into a principle bundle.*

If the underlying manifold M is equipped with a Riemannian metric h, one can construct the orthonormal frame bundle $\mathcal{O}(M)$ as another example of a principal bundle.

Example 2.9 *Let M be a Riemannian manifold and let $x \in M$ be fixed. We call $u\colon \mathbb{R}^d \to T_xM$ a linear isometry if u is a frame at x which satisfies*

$$\langle a, b \rangle = h(ua, ub) \quad \text{for all } a, b \in \mathbb{R}^d \ .$$

Hereby, $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^d$. In particular, if $\{e_1, e_2, \ldots, e_d\}$ is the standard basis of $\mathbb{R}^d$ then $\{ue_i\}_{1 \le i \le d}$ is an orthonormal basis of T_xM. Let $\mathcal{O}(M)_x$ be the set of all linear isometries $u\colon \mathbb{R}^d \to T_xM$ and let

$$\mathcal{O}(M) = \bigcup_{x \in M} \mathcal{O}(M)_x \ .$$

An element $g \in \mathrm{O}(d)$ is a linear isometry $g\colon \mathbb{R}^d \to \mathbb{R}^d$. Hence, $\mathrm{O}(d)$ acts on the right on $\mathcal{O}(M)$ by composition. If we also let $\pi\colon \mathcal{O}(M) \to M$ be the obvious projection map, one can equip $\mathcal{O}(M)$ with a manifold structure which turns $(\mathcal{O}(M), \pi, M, \mathrm{O}(d))$ into a principal bundle.

Before we present the formal definition of a connection on a principal bundle (P, π, M, G), we aim to get an intuitive idea of what a connection gives us. For a fixed $p \in P$, set $x = \pi(p)$ and let $P_x = \pi^{-1}(x)$ be the fibre over x. We observe that T_pP contains the tangent space T_pP_x of the fibre P_x at the point p as a subspace. For obvious reasons, we call T_pP_x the vertical subspace of T_pP and elements of T_pP_x vertical vectors. Moreover, we use V_pP as notation for T_pP_x. Having defined the vertical subspace, we would like to get a notation of horizontal subspace. This is in fact provided by a connection.

Definition 2.10 *A connection on a principal bundle (P, π, M, G) is a smooth map which assigns a subspace $H_pP \subset T_pP$ to each $p \in P$ such that*

(i) $T_pP = V_pP \oplus H_pP$ *, and*

(ii) $H_{p \cdot g}P = (R_g)_* H_pP$ *for every $p \in P$ and $g \in G$.*

Hereby, $(R_g)_*$ denotes the differential of the transformation $R_g\colon P \to P$.

In Kobayashi, Nomizu [12, Chapter 2] it is proved that for the manifolds we consider, a principle bundle always admits a connection. We call H_pP the horizontal subspace of T_pP and elements of H_pP horizontal vectors. However, we need to remember that the horizontal subspace of T_pP does depend on the chosen connection as different connections give rise to different horizontal subspaces.

It remains to discuss the concept of a horizontal lift. Let (P, π, M, G) be a principal bundle with a connection. One easily checks that at every point $p \in P$ the differential of π yields a linear isomorphism $(\pi_*)_p\colon H_pP \to T_{\pi(p)}M$. Thus, for any vector $v \in T_{\pi(p)}M$ there exists a unique horizontal vector $v_p^* \in H_pP$ such that $(\pi_*)_p(v_p^*) = v$. We call v_p^* the horizontal lift of v to p.

This defines all the concepts from Differential Geometry we necessarily need in later parts of the essay. However, for one example in Chapter 4 it is convenient to know how one can use connection forms to obtain horizontal subspaces H_pP satisfying the conditions of Definition 2.10.

Definition 2.11 *A connection form ω on a principal bundle (P, π, M, G) is a 1-form on P taking values in the Lie algebra $\mathfrak{g}$ of G and satisfying the following conditions.*

(i) *For every $g \in G$ and every vector field X on P we have $\omega((R_g)_* X) = \mathrm{Ad}\left(g^{-1}\right)\omega(X)$, where* Ad *denotes the adjoint representation of G in $\mathfrak{g}$.*

(ii) *For every $Y \in \mathfrak{g}$ and $p \in P$ it holds true that $\omega\left(\widetilde{Y}_p\right) = Y$, where*

$$\widetilde{Y}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (p \cdot \exp(tY)) \ .$$

We get a map $\omega_p\colon T_pP \to \mathfrak{g}$ and the horizontal subspace at p is given by $H_pP = \ker(\omega_p)$. In fact, for a given principal bundle, the connections and the connection forms on this bundle are in one-to-one correspondence. For a proof see Kobayashi, Nomizu [12, Chapter 2].

3 Stochastic Calculus on a Riemannian manifold

Throughout this chapter, our stochastic processes X are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and take values in a d-dimensional manifold M. We also allow our processes to be defined only up to some stopping time. However, we generally consider a process X up to the maximum stopping time e for which it can be defined on M. We call e the explosion time of X and write $X = (X_t)_{e>t\geq 0}$.

In Section 3.4, we additionally assume that M is a Riemannian manifold.

3.1 Semimartingales and Stratonovich differentials

Let $f \in C^\infty(\mathbb{R}^d)$ be a smooth function and let $X = (X_t)_{t\geq 0}$ be a semimartingale on $\mathbb{R}^d$. From Itô's formula, it follows that $(f(X_t))_{t\geq 0}$ is a semimartingale on $\mathbb{R}$. This motivates the following definition.

Definition 3.1 *Let $X = (X_t)_{e>t\geq 0}$ be a continuous stochastic process taking values in a manifold M. We call X a semimartingale on M if $(f(X_t))_{e>t\geq 0}$ is a semimartingale on $\mathbb{R}$ for every smooth function $f \in C^\infty(M)$.*

By the above remark, this definition is consistent with our previous definition of a semimartingale on $\mathbb{R}^d$.

From Emery [3, Chapter 1], we recall that if X and Y are semimartingales on $\mathbb{R}$ then the Stratonovich integral of X along Y is defined by

$$\int_0^t X_s \,\partial Y_s = \int_0^t X_s \,\mathrm{d}Y_s + \frac{1}{2}\,[X,Y]_t \ ,$$

where $\int_0^t X_s \,\mathrm{d}Y_s$ is the usual Itô integral of X along Y and $[X,Y]_t$ the quadratic covariation of X and Y. Moreover, if $\widetilde{X}$ is a semimartingale on $\mathbb{R}^d$ and $f \in C^3(\mathbb{R}^d)$ then we have the chain rule

$$f\left(\widetilde{X}_t\right) = f\left(\widetilde{X}_0\right) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x^i}\left(\widetilde{X}_s\right) \partial \widetilde{X}_s^i \ .$$

This is a consequence of Itô's formula, see e.g. Rogers, Williams [16, IV. 46] for the 1-dimensional case, which easily generalises to d dimensions. Using Stratonovich differentials, the chain rule reads

$$\partial f\left(\widetilde{X}_t\right) = \sum_{i=1}^d \frac{\partial f}{\partial x^i}\left(\widetilde{X}_t\right) \partial \widetilde{X}_t^i \ .$$

In the following, we mainly want to use the differential notation. For this, we first need to define the Stratonovich differential $\partial X = (\partial X_t)_{e>t\geq 0}$ of a semimartingale $X = (X_t)_{e>t\geq 0}$ on a manifold M. Hereby, one may think of ∂X as the equivalent of the tangent vector field to a differentiable curve. As in Norris [13] we symbolically define

$$x^i_*(\partial X_t) = \partial(X^i_t) \tag{3.1}$$

for a chart $x = \left(x^1, x^2, \ldots, x^d\right)$ around X_t. However, this symbolic definition has to be understood as part of an integral equation. For instance, if $\alpha \in \Gamma(T^*M)$ is a 1-form on M one can define the integral of α along the semimartingale X in the following way. Let $0 \leq \sigma \leq \tau < e$ be random times with the property that there exists a chart of the manifold M such that $X_t(\omega)$ lies in the domain of that chart for all t satisfying $\sigma(\omega) \leq t \leq \tau(\omega)$. In this chart, we can write

$$\alpha = \sum_{i=1}^d \alpha_i \,\mathrm{d}x^i$$

for smooth functions $a_i \in C^\infty(M)$. The integral of α along X between σ and τ is then defined as

$$\int_\sigma^\tau \alpha_{X_s}(\partial X_s) = \sum_{i=1}^d \int_\sigma^\tau \alpha_i(X_s)\,\partial(X_s^i)\ . \tag{3.2}$$

If $\tilde{x} = \left(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^d\right)$ is another chart satisfying the same restrictions as x subject to σ and τ then

$$\alpha = \sum_{i,j=1}^d \alpha_i \frac{\partial x^i}{\partial \tilde{x}^j}\,\mathrm{d}\tilde{x}^j$$

and

$$\partial(X_t^i) = x_*^i(\partial X_t) = \sum_{j=1}^d \frac{\partial x^i}{\partial \tilde{x}^j}(X_t)\tilde{x}_*^j(\partial X_t) = \sum_{j=1}^d \frac{\partial x^i}{\partial \tilde{x}^j}(X_t)\partial(\widetilde{X}_t^j)$$

by the symbolic definition (3.1). It follows that

$$\sum_{i=1}^d \int_\sigma^\tau \alpha_i(X_s)\,\partial(X_s^i) = \sum_{i,j=1}^d \int_\sigma^\tau \alpha_i(X_s)\frac{\partial x^i}{\partial \tilde{x}^j}(X_s)\partial(\widetilde{X}_s^j) = \sum_{j=1}^d \int_\sigma^\tau \tilde{\alpha}_j(X_s)\,\partial(\widetilde{X}_s^i)\ .$$

Therefore, (3.2) is independent of the chosen chart and by patching the integral together across overlapping charts, we can define

$$\int_0^t \alpha_{X_s}(\partial X_s)$$

for $e > t \geq 0$. For more details see Emery [3, Chapter 7]. It is a pleasant feature that the chain rule for the Stratonovich integral extends to functions on a manifold M.

Lemma 3.2 *Let M be a manifold, let $f \in C^\infty(M)$ be a smooth function and let X be a semimartingale on M. It holds true that*

$$\partial f(X_t) = (\mathrm{d}f)_{X_t}(\partial X_t)\ .$$

Proof. We recall that this identity has to be understood as

$$f(X_t) = f(X_0) + \int_0^t (\mathrm{d}f)_{X_s}(\partial X_s)\ .$$

Let $0 \leq \sigma \leq \tau < e$ be random times such that there exists a chart x of M with the same properties as above. By (3.2), we have

$$\int_\sigma^\tau (\mathrm{d}f)_{X_s}(\partial X_s) = \sum_{i=1}^d \int_\sigma^\tau \frac{\partial f}{\partial x^i}\,(X_s)\ \partial(X_s^i)\ .$$

Furthermore, the chain rule for semimartingales on $\mathbb{R}^d$ yields

$$\sum_{i=1}^d \int_\sigma^\tau \frac{\partial f}{\partial x^i}(X_s)\,\partial(X_s^i) = \left(f \circ x^{-1}\right)(x(X_\tau)) - \left(f \circ x^{-1}\right)(x(X_\sigma)) = f(X_\tau) - f(X_\sigma)\ .$$

Patching integrals together across overlapping charts gives the desired result. ☐

3.2 Stochastic differential equations

Having defined Stratonovich differentials, we can now explain what we mean by a stochastic differential equation (SDE), cf. Emery [3, Chapter 7]. Let M_1, M_2 be manifolds, let $V \in \Gamma(T^*M_1 \otimes TM_2)$ be a smooth section over $M_1 \times M_2$ and let $X = (X_t)_{e>t\geq 0}$ be a semimartingale on M_1. We say that a semimartingale $Y = (Y_t)_{\zeta>t\geq 0}$ on M_2 with $e \geq \zeta$ is a solution of the stochastic differential equation

$$\partial Y_t = V(X_t, Y_t)\, \partial X_t \tag{3.3}$$

if for every 1-form $\alpha \in \Gamma(T^*M_2)$

$$\int_0^t \alpha_{Y_s}(\partial Y_s) = \int_0^t \alpha_{Y_s}\left(V(X_s, Y_s)\, \partial X_s\right)$$

holds true for all $t < \zeta$. As stated in Emery [3, Theorem 7.21], we have the following existence and uniqueness property.

Theorem 3.3 *Suppose we are given two manifolds M_1 and M_2, a section $V \in \Gamma(T^*M_1 \otimes TM_2)$, a semimartingale X on M_1 and an $\mathcal{F}_0$-measurable random variable Y_0 on M_2. Then there exists a stopping time ζ and a semimartingale $Y = (Y_t)_{\zeta>t\geq 0}$ on M_2 starting from Y_0 with the following properties.*

(i) *Y is a solution of the stochastic differential equation* (3.3).

(ii) *If ζ is finite then Y explodes at ζ, i.e. for almost every $\omega \in \Omega$ the path $(Y_t(\omega))_{\zeta>t\geq 0}$ is not contained in any compact subset of M_2.*

(iii) *If $\widetilde{Y} = \left(\widetilde{Y}_t\right)_{\widetilde{\zeta}>t\geq 0}$ is another solution of* (3.3) *starting from $\widetilde{Y}_0 = Y_0$ then almost surely $\widetilde{\zeta} \leq \zeta$ and almost surely $\widetilde{Y}_t = Y_t$ for all $0 \leq t < \widetilde{\zeta}$.*

We call Y the unique solution of (3.3) up to explosion.

3.3 Quadratic variation of a semimartingale

We would like to have the notion of a quadratic variation process associated to a semimartingale X on the manifold M. In fact, it turns out to be convenient to define a quadratic variation process for each tensor field $b \in \Gamma(T^*M \otimes T^*M)$.

Let X be a semimartingale on M. In Emery [3, Theorem 3.8] it is established that there exists a unique $\mathbb{R}$-linear map, denoted by

$$b \longmapsto \int b(\mathrm{d}X, \mathrm{d}X)\ ,$$

from $\Gamma(T^*M \otimes T^*M)$ to the space of real-valued continuous stochastic processes with finite variation such that for all smooth functions $f, g \in C^\infty(M)$ we have

(i) $\int (fb)(\mathrm{d}X, \mathrm{d}X) = \int f(X)\, \mathrm{d}\left(\int b(dX, dX)\right)$, and

(ii) $\int (\mathrm{d}f \otimes \mathrm{d}g)(\mathrm{d}X, \mathrm{d}X) = [f(X), g(X)]$.

We call $\int b(\mathrm{d}X, \mathrm{d}X)$ the b-quadratic variation of X and denote its value at t by $\int_0^t b(\mathrm{d}X_s, \mathrm{d}X_s)$.

3.4 Horizontal lift and stochastic development

Let M be a Riemannian manifold of dimension d and let $\mathcal{O}(M)$ be the orthonormal frame bundle over M. To be able to talk about horizontal vectors in $T\mathcal{O}(M)$ we need to choose a connection on $\mathcal{O}(M)$. One can show that there is a one-to-one correspondence between metric covariant derivatives on M and connections on $\mathcal{O}(M)$, cf. Kobayashi, Nomizu [12, Chapter 4]. From now onwards, we shall equip $\mathcal{O}(M)$ with the connection which corresponds to the Levi-Civita connection on M.

In Differential Geometry, we have the following notions. A differentiable curve $u = (u_t)_{t\in[a,b]}$ on $\mathcal{O}(M)$ is called a horizontal curve if all its tangent vectors are horizontal. Moreover, if $x = (x_t)_{t\in[a,b]}$ is a differentiable curve on M and $u_a \in \mathcal{O}(M)$ is a fixed frame at x_a then a horizontal curve $u = (u_t)_{t\in[a,b]}$ on $\mathcal{O}(M)$ starting from u_a with $\pi(u) = x$ is called horizontal lift of x to u_a. In fact, there is a unique curve u satisfying these conditions, cf. Kobayashi, Nomizu [12, Chapter 2]. By the chain rule, we also have

$$\dot{x}_t = (\pi_*)_{u_t}(\dot{u}_t) \ ,$$

where $\dot{x}_t$ and $\dot{u}_t$ are the tangent vectors to x and u at x_t and u_t, respectively. Furthermore, the anti-development $w = (w_t)_{t\in[a,b]}$ of the curve x is defined by

$$w_t = \int_a^t u_s^{-1}\dot{x}_s \,\mathrm{d}s$$

for $a \leq t \leq b$. We note that w is a curve on $\mathbb{R}^d$ with $w_a = 0$.

If we are given a semimartingale $X = (X_t)_{e>t\geq 0}$ on M we would like to talk about its horizontal lift $U = (U_t)_{e>t\geq 0}$ to $\mathcal{O}(M)$ as well as its anti-development $W = (W_t)_{e>t\geq 0}$ on $\mathbb{R}^d$. The problem is that a path $X(\omega)$ of X for some $\omega \in \Omega$ need not be differentiable. Therefore, we cannot use the notions of horizontal lift and anti-development as defined above. A definition which works for semimartingales makes use of the following horizontal vector fields. Let $v \in \mathbb{R}^d$ be fixed. We define $H_v \in \Gamma(H\mathcal{O}(M))$ by $H_v(u) = (uv)_u^*$ for $u \in \mathcal{O}(M)$, i.e. $H_v(u)$ is the unique horizontal lift of $uv \in T_{\pi u}M$ to u. Moreover, if $\{e_i\}_{1\leq i\leq d}$ denotes the standard basis of $\mathbb{R}^d$ we set $H_i = H_{e_i}$. We observe that at each $u \in \mathcal{O}(M)$ the horizontal vectors $H_1(u), H_2(u), \ldots, H_d(u)$ form a basis of $H_u\mathcal{O}(M)$. In particular, if $u = (u_t)_{t\in[a,b]}$ is a horizontal curve on $\mathcal{O}(M)$ we can find curves $\left(\alpha_t^1\right)_{t\in[a,b]}, \left(\alpha_t^2\right)_{t\in[a,b]}, \ldots, \left(\alpha_t^d\right)_{t\in[a,b]}$ on $\mathbb{R}$ such that

$$\dot{u}_t = \sum_{i=1}^d H_i(u_t)\alpha_t^i \ .$$

If we let $w_t^i = \int_a^t \alpha_s^i \,\mathrm{d}s$, then

$$\dot{u}_t = \sum_{i=1}^d H_i(u_t)\dot{w}_t^i \ .$$

In fact, if u is the horizontal lift of a curve x on M then $w = \left(w^1, w^2, \ldots, w^d\right)$ is indeed the anti-development of x. The latter follows from

$$\dot{x}_t = (\pi_*)_{u_t}(\dot{u}_t) = (\pi_*)_{u_t}\left(\sum_{i=1}^d H_i(u_t)\right)\alpha_t^i = \sum_{i=1}^d (u_t e_i)\,\alpha_t^i = u_t\left(\sum_{i=1}^d e_i\alpha_t^i\right)$$

as it gives

$$u_t^{-1}\dot{x}_t = \left(\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^d\right) \ .$$

This motivates the following definitions, cf. Hsu [9, Chapter 2.3].

Definition 3.4 *A semimartingale U on $\mathcal{O}(M)$ is called horizontal if there exists a semimartingale W on $\mathbb{R}^d$ such that*

$$\partial U_t = \sum_{i=1}^{d} H_i(U_t)\,\partial W_t^i \,. \tag{3.4}$$

By construction, $(H_1, H_2, \ldots, H_d)$ is a global frame of the horizontal bundle $H\mathcal{O}(M)$. Let $(H^1, H^2, \ldots, H^d)$ denote its dual frame. If U is a horizontal semimartingale on $\mathcal{O}(M)$ then Equation (3.4) implies

$$H^j(U_t)\,\partial U_t = \sum_{i=1}^{d} H^j(U_t)H_i(U_t)\,\partial W_t^i = \partial W_t^j$$

for $1 \le j \le d$. Hence, if we fix W_0 then W is uniquely given by

$$W_t^j = W_0^j + \int_0^t H^j(U_s)\,\partial U_s \,.$$

Definition 3.5 *A horizontal lift U of a semimartingale X on M is a horizontal semimartingale on $\mathcal{O}(M)$ such that $\pi(U) = X$. Moreover, the unique semimartingale W on $\mathbb{R}^d$ satisfying* (3.4) *and $W_0 = 0$ is called an anti-development of X.*

In Hsu [9, Chapter 2.3], it is proved that for a given semimartingale $X = (X_t)_{e>t\ge 0}$ on M and an $\mathcal{F}_0$-measurable random variable U_0 on $\mathcal{O}(M)$ with $\pi(U_0) = X_0$ there exists a unique horizontal lift $U = (U_t)_{e>t\ge 0}$ starting from U_0. In fact, Hsu does not restrict his attention to $\mathcal{O}(M)$ until [9, Chapter 3]. Before that, he still works with the frame bundle $\mathscr{F}(M)$. However, one can consider $\mathcal{O}(M)$ as a subbundle of $\mathscr{F}(M)$. If one then equips $\mathscr{F}(M)$ with a connection which corresponds to a metric covariant derivative on M and ensures that U_0 takes values in $\mathcal{O}(M)$ then the unique horizontal lift $U = (U_t)_{e>t\ge 0}$ to $\mathscr{F}(M)$ stays in $\mathcal{O}(M)$. Thus, we also get uniqueness for the situation we consider.

Conversely, if W is a semimartingale on $\mathbb{R}^d$ and U_0 is an $\mathcal{F}_0$-measurable random variable on $\mathcal{O}(M)$, then by Theorem 3.3 applied to

$$V = \sum_{i=1}^{d} \mathrm{d}x^i \otimes H_i \in \Gamma\left(T^*\mathbb{R}^d \otimes T\mathcal{O}(M)\right) ,$$

there exists a unique semimartingale $U = (U_t)_{e>t\ge 0}$ starting from U_0 and satisfying (3.4) up to explosion. We call $X = (X_t)_{e>t\ge 0}$ given by $\pi(U) = X$ the development of W on M. Moreover, X is independent of the choice of the initial frame U_0 over X_0, see Elworthy [2, Lemma 11A].

We use the correspondence between semimartingales X on M and semimartingales W on $\mathbb{R}^d$ in Section 4.1. There we characterise Brownian motion on Riemannian manifolds using stochastic differential equations.

4 Equivalent definitions of Brownian motion

In this chapter, we discuss various characterisations of Brownian motion on a Riemannian manifold. To begin with, we need to agree on one definition and similar to Emery [3, Chapter 5] we make the following definition.

Definition 4.1 *Let M be a Riemannian manifold, let $X = (X_t)_{e>t\geq 0}$ be a continuous M-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\left(\mathcal{F}_t^X\right)_{e>t\geq 0}$ denote the natural filtration generated by the process X. We say that X is a Brownian motion on M if for all smooth functions $f \in C^\infty(M)$ the process $N = (N_t)_{e>t\geq 0}$ given by*

$$N_t = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_M f(X_s)\,\mathrm{d}s$$

is a local martingale on $\mathbb{R}$ in the filtration $\left(\mathcal{F}_t^X\right)_{e>t\geq 0}$.

Our first observation about Brownian motion is stated in the next lemma.

Lemma 4.2 *If X is a Brownian motion on a Riemannian manifold M then X is a semimartingale on M in the natural filtration $\left(\mathcal{F}_t^X\right)_{e>t\geq 0}$.*

Proof. It suffices to note that for all $f \in C^\infty(M)$ the process $A = (A_t)_{e>t\geq 0}$ defined by

$$A_t = \frac{1}{2}\int_0^t \Delta_M f(X_s)\,\mathrm{d}s$$

is an adapted process of finite variation. □

For $M = \mathbb{R}^d$ equipped with the Euclidean metric, Definition 4.1 has to be consistent with Definition 1.1. Indeed, this is ensured by Proposition 1.5 since $\Delta_{\mathbb{R}^d} = \Delta$.

As remarked in Hsu [9, Chapter 3.2], sometimes it is necessary to extend our definition of Brownian motion slightly. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let X be a Brownian motion according to Definition 4.1. If X is additionally adapted to $(\mathcal{F}_t)_{t\geq 0}$ and satisfies the strong Markov property with respect to that filtration then we say that X is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$. We shall work in this set-up from here onwards.

Having discussed the definition of Brownian motion on a Riemannian manifold we present the first alternative characterisation, cf. Proposition 4.5. This one is taken from Emery [3, Chapter 5]. It needs one more definition and one preliminary lemma. Both of them make use of the b-quadratic variation, and particularly of the $\nabla^2 f$-quadratic variation, defined in Section 3.3.

Definition 4.3 *Let M be a Riemannian manifold equipped with the Levi-Civita connection ∇. A semimartingale $X = (X_t)_{e>t\geq 0}$ on M is called a martingale if*

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s)$$

is a local martingale on $\mathbb{R}$ for every smooth function $f \in C^\infty(M)$.

We remark that this is standard terminology, even though it would seem more sensible to call the martingales we have just defined 'local martingales'. In particular, a martingale on $\mathbb{R}^d$ according to Definition 4.3 is in fact a local martingale. However, we only consider martingales according to the new definition up to and including Section 4.1, whereas we do not need true martingales on $\mathbb{R}^d$ according to the usual definition before Section 4.3. Therefore, the terminology just introduced should not cause any confusions.

Lemma 4.4 *Let M be a Riemannian manifold with metric h. If $X = (X_t)_{e>t\geq 0}$ is a semimartingale on M satisfying*

$$[f(X), f(X)]_t = \int_0^t h\left(\operatorname{grad} f, \operatorname{grad} f\right)(X_s)\,\mathrm{d}s \tag{4.1}$$

for every smooth function $f \in C^\infty(M)$ then

$$\int_0^t b(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t \operatorname{Tr}(b)(X_s)\,\mathrm{d}s$$

for every smooth tensor field $b \in \Gamma(T^*M \otimes T^*M)$.

Proof. Let $f, g \in C^\infty(M)$ be arbitrary. By polarising Equation (4.1), one obtains

$$[f(X), g(X)]_t = \int_0^t h\left(\operatorname{grad} f, \operatorname{grad} g\right)(X_s)\,\mathrm{d}s\ . \tag{4.2}$$

Moreover, let $(E_1, E_2, \ldots, E_n)$ be a local orthonormal frame. We deduce that

$$\begin{aligned}\operatorname{Tr}(\mathrm{d}f \otimes \mathrm{d}g) &= \sum_{i=1}^d \mathrm{d}f(E_i)\mathrm{d}g(E_i) = \sum_{i=1}^d h\left(\operatorname{grad} f, E_i\right) h\left(\operatorname{grad} g, E_i\right) \\ &= h\left(\operatorname{grad} f, \operatorname{grad} g\right)\end{aligned}$$

by Parseval's identity. From (4.2), it follows that

$$[f(X), g(X)]_t = \int_0^t \operatorname{Tr}(\mathrm{d}f \otimes \mathrm{d}g)(X_s)\,\mathrm{d}s\ .$$

By property (ii) of the quadratic variation process, this implies

$$\int_0^t \left(\mathrm{d}f \otimes \mathrm{d}g\right)\left(\mathrm{d}X_s, \mathrm{d}X_s\right) = \int_0^t \operatorname{Tr}(\mathrm{d}f \otimes \mathrm{d}g)(X_s)\,\mathrm{d}s\ . \tag{4.3}$$

Furthermore, under our assumptions on the manifold M every smooth $b \in \Gamma(T^*M \otimes T^*M)$ can be written as

$$b = \sum_{i,j=1}^d b_{ij}\,\mathrm{d}f^i \otimes \mathrm{d}f^j$$

for smooth functions $b_{ij}, f^i, f^j \in C^\infty(M)$, see Emery [3, Lemma 2.23]. Hence, from (4.3) and property (i) of the quadratic variation process, we conclude

$$\begin{aligned}\int_0^t b(\mathrm{d}X_s, \mathrm{d}X_s) &= \sum_{i,j=1}^d \int_0^t b_{ij}(X_s)\,\mathrm{d}\left(\int_0^s \left(\mathrm{d}f^i \otimes \mathrm{d}f^j\right)\left(\mathrm{d}X_r, \mathrm{d}X_r\right)\right) \\ &= \sum_{i,j=1}^d \int_0^t b_{ij}(X_s)\operatorname{Tr}(\mathrm{d}f^i \otimes \mathrm{d}f^j)(X_s)\,\mathrm{d}s \\ &= \int_0^t \operatorname{Tr}(b)(X_s)\,\mathrm{d}s\ ,\end{aligned}$$

as claimed. □

We use the preceding lemma to prove the next proposition.

Proposition 4.5 *Let M be a Riemannian manifold. A semimartingale $X = (X_t)_{e>t\geq 0}$ on M is a Brownian motion if and only if X is a martingale on M and for every smooth function $f \in C^\infty(M)$ we have*

$$[f(X), f(X)]_t = \int_0^t h\left(\operatorname{grad} f, \operatorname{grad} f\right)(X_s)\,\mathrm{d}s\,. \tag{4.4}$$

Proof. Let $f \in C^\infty(M)$ be arbitrary. For the 'if' direction, we observe that by Lemma 4.4 applied to $\nabla^2 f$ and by Lemma 2.4

$$\int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t \operatorname{Tr}\left(\nabla^2 f\right)(X_s)\,\mathrm{d}s = \int_0^t \Delta_M f(X_s)\,\mathrm{d}s\,.$$

As X is a martingale by assumption, this further implies that

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_M f(X_s)\,\mathrm{d}s = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s) \tag{4.5}$$

is a local martingale on $\mathbb{R}$. Thus, X is indeed a Brownian motion on M.
Conversely, assume that X is a Brownian motion on M. From

$$\Delta_M\left(f^2\right) = 2f\Delta_M f + 2h(\operatorname{grad} f, \operatorname{grad} f)$$

and the definition of Brownian motion, we deduce that

$$f^2(X_t) - f^2(X_0) - \int_0^t f(X_s)\Delta_M f(X_s) - \int_0^t h(\operatorname{grad} f, \operatorname{grad} f)(X_s)\,\mathrm{d}s \tag{4.6}$$

is a local martingale. Moreover, if Y is a semimartingale on $\mathbb{R}$ then by Itô's formula

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s\,\mathrm{d}Y_s + [Y, Y]_t\,.$$

Applying this to the real-valued semimartingale $Y = (f(X_t))_{e>t\geq 0}$ gives

$$f^2(X_t) = f^2(X_0) + 2\int_0^t f(X_s)\,\mathrm{d}f(X_s) + [f(X), f(X)]_t\,. \tag{4.7}$$

By again using the fact that X is a Brownian motion on M, we also know that the process $N = (N_t)_{e>t\geq 0}$ given by

$$N_t = f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_M f(X_s)\,\mathrm{d}s$$

is a local martingale on $\mathbb{R}$. Since the Itô integral preserves local martingales, it follows that

$$\int_0^t f(X_s)\,\mathrm{d}N_s = \int_0^t f(X_s)\,\mathrm{d}f(X_s) - \frac{1}{2}\int_0^t f(X_s)\Delta_M f(X_s)\,\mathrm{d}s$$

is also a local martingale. Substituting this into Equation (4.7) yields that

$$f^2(X_t) - f^2(X_0) - \int_0^t f(X_s)\Delta_M f(X_s)\,\mathrm{d}s - [f(X), f(X)]_t$$

is a local martingale on $\mathbb{R}$. From (4.6) we therefore deduce that

$$\int_0^t h(\operatorname{grad} f, \operatorname{grad} f)(X_s)\,\mathrm{d}s - [f(X), f(X)]_t$$

is a local martingale. On the other hand, it is also a process of finite variation and hence, it must be constant. As it takes the value zero at $t = 0$, it follows that

$$\int_0^t h(\operatorname{grad} f, \operatorname{grad} f)(X_s)\,\mathrm{d}s - [f(X), f(X)]_t = 0\ ,$$

which establishes (4.4). In particular, we are now able to apply Lemma 4.4. As before, we obtain

$$\int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t \Delta_M f(X_s)\,\mathrm{d}s$$

as well as Equality (4.5). Since X is a Brownian motion on M this implies that X is a martingale on M. □

We observe the similarities between Lévy's characterisation of Brownian motion and Proposition 4.5. Both criterions characterise Brownian motion as a (local) martingale with a certain quadratic variation process.

4.1 Characterising Brownian motion using SDEs

In this section, we present the construction of Brownian motion which is due to Eells, Elworthy and Malliavin, cf. Theorem 4.7. We use the same set-up as in Section 3.4.

To prove Theorem 4.7, we need the following lemma.

Lemma 4.6 *If X is a semimartingale on a Riemannian manifold M and $f \in C^\infty(M)$ then*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t H_i\tilde{f}(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t H_iH_j\tilde{f}(U_s)\,\mathrm{d}\left[W^i, W^j\right]_s\ ,$$

where $\tilde{f} = f \circ \pi \colon \mathcal{O}(M) \to \mathbb{R}$.

Proof. Using the chain rule for Stratonovich differentials, cf. Lemma 3.2, and the Relation (3.4) we obtain

$$\partial\tilde{f}(U_t) = (\mathrm{d}\tilde{f})_{U_t}(\partial U_t) = \sum_{i=1}^d (\mathrm{d}\tilde{f})_{U_t}\,(H_i(U_t))\,\partial W_t^i\ .$$

By definition, it holds true that $(\mathrm{d}\tilde{f})_{U_t}\,(H_i(U_t)) = H_i\tilde{f}(U_t)$ for all $1 \le i \le d$, which implies

$$\tilde{f}(U_t) - \tilde{f}(U_0) = \sum_{i=1}^d \int_0^t H_i\tilde{f}(U_s)\,\partial W_s^i\ .$$

Since $\pi(U_t) = X_t$, we have $\tilde{f}(U_t) - \tilde{f}(U_0) = f(X_t) - f(X_0)$. Moreover, as W is a semimartingale on $\mathbb{R}^d$ we can write the Stratonovich integral in terms of an Itô integral. This yields

$$f(X_t) - f(X_0) = \sum_{i=1}^d \int_0^t H_i\tilde{f}(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i=1}^d \left[H_i\tilde{f}(U), W^i\right]_t\ . \tag{4.8}$$

By applying the chain rule for Stratonovich differentials to the function $H_i\tilde{f} \colon \mathcal{O}(M) \to \mathbb{R}$ we also get

$$\begin{aligned}\partial(H_i\tilde{f})(U_t) &= \left(\mathrm{d}(H_i\tilde{f})\right)_{U_t}(\partial U_t) = \sum_{j=1}^d \left(\mathrm{d}(H_i\tilde{f})\right)_{U_t}(H_j(U_t))\,\partial W_t^j \\ &= \sum_{j=1}^d H_jH_i\tilde{f}(U_t)\,\partial W_t^j\end{aligned}$$

for all $1 \le i \le d$. Since the Itô and the Stratonovich integral only differ by a finite variation term, we deduce that

$$\left[H_i\tilde{f}(U), W^i\right]_t = \sum_{j=1}^{d} \left[\int H_j H_i \tilde{f}(U)\,\mathrm{d}W^j, W^i\right]_t \ .$$

By the Kunita-Watanabe identity, see e.g. Rogers, Williams [16, IV. 28], it follows that

$$\left[H_i\tilde{f}(U), W^i\right]_t = \sum_{j=1}^{d} \int_0^t H_j H_i \tilde{f}(U_s)\,\mathrm{d}\left[W^j, W^i\right]_s \ .$$

From (4.8) we then obtain

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t H_i\tilde{f}(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t H_i H_j \tilde{f}(U_s)\,\mathrm{d}\left[W^i, W^j\right]_s \ ,$$

as claimed. □

For the most part, the proof of the next theorem follows Hsu [9, Chapter 3.2]. However, one intermediate result is deduced differently as we have Lemma 4.4 and Proposition 4.5 available.

Theorem 4.7 *Let M be a Riemannian manifold of dimension d and let X be a semimartingale on M. Then X is a Brownian motion on M if and only if its anti-development W is a standard Brownian motion on $\mathbb{R}^d$.*

Proof. For the 'if' direction, let $f \in C^\infty(M)$ be arbitrary and set $\tilde{f} = f \circ \pi$. By assumption, W is a standard Brownian motion on $\mathbb{R}^d$ and Lévy's characterisation implies that $\left[W^i, W^j\right]_t = t\delta^{ij}$ for $1 \le i, j \le d$. From Lemma 4.6, it follows that

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t H_i\tilde{f}(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i=1}^{d} \int_0^t H_i^2 \tilde{f}(U_s)\,\mathrm{d}s \ . \tag{4.9}$$

In the last term, one recognises Bochner's horizontal Laplacian

$$\Delta_{\mathcal{O}(M)} = \sum_{i=1}^{d} H_i^2 \ .$$

As proved in Hsu [9, Chapter 3.1], we have

$$\Delta_{\mathcal{O}(M)}\tilde{f}(u) = \Delta_M f(\pi u) \tag{4.10}$$

for any $u \in \mathcal{O}(M)$. Thus, Equation (4.9) implies that

$$\begin{aligned} f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_M f(X_s)\,\mathrm{d}s &= f(X_t) - f(X_0) - \frac{1}{2}\sum_{i=1}^{d} \int_0^t H_i^2 \tilde{f}(U_s)\,\mathrm{d}s \\ &= \sum_{i=1}^{d} \int_0^t H_i\tilde{f}(U_s)\,\mathrm{d}W_s^i \ . \end{aligned}$$

The latter is indeed a local martingale as W is a local martingale by Lévy's characterisation. Hence, X is a Brownian motion on M, as claimed.

It remains to prove the 'only if' direction. By the Nash embedding theorem, we may assume that the Riemannian manifold M is isometrically embedded into Euclidean space $\mathbb{R}^N$ for some $N \in \mathbb{N}$. Let $\{\eta_\alpha\}_{1\le\alpha\le N}$ be the standard basis of $\mathbb{R}^N$ and let $P_1, P_2, \dots, P_N$ be the vector fields on M which one obtains by defining $(P_\alpha)_x$ to be the orthogonal projection of η_α onto T_xM. In the following, we use the coordinate functions $f^\alpha\colon M \to \mathbb{R}$ given by $f^\alpha(x) = x^\alpha$ where $x = (x^1, x^2, \dots, x^N)$ considered as an element of $\mathbb{R}^N$. Set $X_t^\alpha = f^\alpha(X_t)$ and note that $f^\alpha \in C^\infty(M)$ for each $1 \le \alpha \le N$. As we assume that X is a Brownian motion, it follows that

$$N_t^\alpha = X_t^\alpha - X_0^\alpha - \frac{1}{2}\int_0^t \Delta_M f^\alpha(X_s)\,\mathrm{d}s \tag{4.11}$$

is a local martingale. On the other hand, by applying Lemma 4.6 to the function f^α we also obtain

$$X_t^\alpha = X_0^\alpha + \sum_{i=1}^d \int_0^t H_i\tilde{f}^\alpha(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t H_iH_j\tilde{f}^\alpha(U_s)\,\mathrm{d}\left[W^i, W^j\right]_s\ . \tag{4.12}$$

Furthermore, for any $f \in C^\infty(M)$ it holds true that

$$\begin{aligned}\sum_{i,j=1}^d \int_0^t H_iH_j\tilde{f}(U_s)\,\mathrm{d}\left[W^i, W^j\right]_s &= \sum_{i,j=1}^d \int_0^t \nabla^2 f(U_se_i, U_se_j)\,\mathrm{d}\left[W^i, W^j\right]_s \\ &= \int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s)\ ,\end{aligned}$$

where $\{e_1, e_2, \dots, e_d\}$ is the standard basis of $\mathbb{R}^d$. The result is a consequence of the facts $H_iH_j\tilde{f}(u) = \nabla^2 f(ue_i, ue_j)$ for any $f \in C^\infty(M)$, cf. Hsu [9, Chapter 2.2], and

$$\int_0^t b(\mathrm{d}X_s, \mathrm{d}X_s) = \sum_{i,j=1}^d \int_0^t b(U_se_i, U_se_j)\,\mathrm{d}\left[W^i, W^j\right]_s \tag{4.13}$$

for any $b \in \Gamma(T^*M \otimes T^*M)$, see Emery [3, Lemma 8.25]. Moreover, since X is a Brownian motion by assumption, we can use Proposition 4.5 and Lemma 4.4 to deduce that

$$\int_0^t \nabla^2 f(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t \Delta_M f(X_s)\,\mathrm{d}s\ ,$$

for any $f \in C^\infty(M)$. In particular, it follows that

$$\sum_{i,j=1}^d \int_0^t H_iH_j\tilde{f}^\alpha(U_s)\,\mathrm{d}\left[W^i, W^j\right]_s = \int_0^t \Delta_M f^\alpha(X_s)\,\mathrm{d}s\ .$$

Substituting this into Equation (4.12) yields

$$X_t^\alpha = X_0^\alpha + \sum_{i=1}^d \int_0^t H_i\tilde{f}^\alpha(U_s)\,\mathrm{d}W_s^i + \frac{1}{2}\int_0^t \Delta_M f^\alpha(X_s)\,\mathrm{d}s$$

and from (4.11) we then obtain

$$N_t^\alpha = \sum_{i=1}^d \int_0^t H_i\tilde{f}^\alpha(U_s)\,\mathrm{d}W_s^i\ .$$

We claim that $H_i\tilde{f}^\alpha(u) = \langle \eta_\alpha, ue_i\rangle$ for any $u \in \mathcal{O}(M)$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^N$. To prove the claim, we observe that if $v^* \in H_u\mathcal{O}(M)$ and $v = (\pi_*)_u(v^*)$ then

$$v^*\tilde{f} = \mathrm{d}\tilde{f}(v^*) = \mathrm{d}(f\circ\pi)(v^*) = \mathrm{d}f\left((\pi_*)_u(v^*)\right) = \mathrm{d}f(v) = h(\operatorname{grad} f, v)$$

for $f \in C^\infty(M)$ and h the Riemannian metric on M. Since M is isometrically embedded into $\mathbb{R}^N$, we further have that $h(\operatorname{grad} f, v) = \langle \operatorname{grad} f, v\rangle$ and therefore,

$$v^* \tilde{f} = \langle \operatorname{grad} f, v\rangle \ .$$

Applying this general result to the horizontal lift $H_i(u)$ of ue_i to u and the function f^α as well as noting that $\operatorname{grad} f^\alpha = P_\alpha$ yields $H_i\tilde{f}^\alpha(u) = \langle P_\alpha, ue_i\rangle$. The claimed result

$$H_i\tilde{f}^\alpha(u) = \langle \eta_\alpha, ue_i\rangle$$

follows because $(P_\alpha)_{\pi u}$ is the orthogonal projection of η_α onto $T_{\pi u}M$ and $ue_i \in T_{\pi u}M$. Multiplying

$$\mathrm{d}N_t^\alpha = \sum_{i=1}^d \langle \eta_\alpha, U_t e_i\rangle \, \mathrm{d}W_t^i$$

by $\langle \eta_\alpha, U_t e_j\rangle$, summing over α and using Parseval's identity as well as the orthonormality of the vectors $U_t e_1, U_t e_2, \dots, U_t e_d$ gives

$$\sum_{\alpha=1}^N \langle \eta_\alpha, U_t e_j\rangle \, \mathrm{d}N_t^\alpha = \sum_{\alpha=1}^N \sum_{i=1}^d \langle \eta_\alpha, U_t e_j\rangle \langle \eta_\alpha, U_t e_i\rangle \, \mathrm{d}W_t^i = \sum_{i=1}^d \langle U_t e_j, U_t e_i\rangle \, \mathrm{d}W_t^i = \mathrm{d}W_t^j \ .$$

Since $W_0 = 0$, we have established that

$$W_t^j = \sum_{\alpha=1}^N \int_0^t \langle \eta_\alpha, U_s e_j\rangle \, \mathrm{d}N_s^\alpha. \tag{4.14}$$

This implies that W is a local martingale on $\mathbb{R}^d$ as N^α is a local martingale on $\mathbb{R}$ for each α. In the following, we compute the quadratic variation of W. For $1 \le \alpha, \beta \le N$, we deduce from (4.11) that

$$\left[N^\alpha, N^\beta\right]_t = \left[X^\alpha, X^\beta\right]_t \tag{4.15}$$

since $\int_0^t \Delta_M f^\alpha(X_s) \, \mathrm{d}s$ is a term of finite variation. Due to the integration by parts formula for real-valued semimartingales we have

$$X_t^\alpha X_t^\beta = X_0^\alpha X_0^\beta + \int_0^t X_s^\alpha \, \mathrm{d}X_s^\beta + \int_0^t X_s^\beta \, \mathrm{d}X_s^\alpha + \left[X^\alpha, X^\beta\right]_t \ .$$

From (4.11), we also obtain $\mathrm{d}X_t^\alpha = \mathrm{d}N_t^\alpha + \frac{1}{2}\Delta_M f^\alpha(X_t) \, \mathrm{d}t$. Substituting this into the previous formula yields

$$\begin{aligned} X_t^\alpha X_t^\beta = X_0^\alpha X_0^\beta &+ \int_0^t X_s^\alpha \, \mathrm{d}N_s^\beta + \frac{1}{2}\int_0^t X_s^\alpha \Delta_M f^\beta(X_s) \, \mathrm{d}s \\ &+ \int_0^t X_s^\beta \, \mathrm{d}N_s^\alpha + \frac{1}{2}\int_0^t X_s^\beta \Delta_M f^\alpha(X_s) \, \mathrm{d}s + \left[X^\alpha, X^\beta\right]_t \ . \end{aligned} \tag{4.16}$$

On the other hand, by considering the function $f^{\alpha\beta} = f^\alpha f^\beta \colon M \to \mathbb{R}$ and by using the assumption that X is a Brownian motion, we know that

$$N_t^{\alpha\beta} = X_t^\alpha X_t^\beta - X_0^\alpha X_0^\beta - \frac{1}{2}\int_0^t \Delta_M f^{\alpha\beta}(X_s) \, \mathrm{d}s$$

is a local martingale. Moreover, it holds true that

$$\Delta_M f^{\alpha\beta} = \Delta_M \left(f^\alpha f^\beta\right) = \left(\Delta_M f^\alpha\right) f^\beta + f^\alpha \left(\Delta_M f^\beta\right) + 2h\left(\operatorname{grad} f^\alpha, \operatorname{grad} f^\beta\right) \ .$$

Since $h\left(\operatorname{grad} f^\alpha, \operatorname{grad} f^\beta\right) = \langle P_\alpha, P_\beta\rangle$ it follows that

$$X_t^\alpha X_t^\beta = X_0^\alpha X_0^\beta + N_t^{\alpha\beta} + \frac{1}{2}\int_0^t X_s^\alpha \Delta_M f^\beta(X_s)\,\mathrm{d}s + \frac{1}{2}\int_0^t X_s^\beta \Delta_M f^\alpha(X_s)\,\mathrm{d}s + \int_0^t \langle P_\alpha, P_\beta\rangle(X_s)\,\mathrm{d}s\ .$$

Comparing this with (4.16) gives

$$N_t^{\alpha\beta} - \int_0^t X_s^\alpha\,\mathrm{d}N_s^\beta - \int_0^t X_s^\beta\,\mathrm{d}N_s^\alpha = \left[X^\alpha, X^\beta\right]_t - \int_0^t \langle P_\alpha, P_\beta\rangle(X_s)\,\mathrm{d}s\ .$$

We note that the left-hand side is a local martingale, whereas the right-hand side is a finite variation process starting from zero. Thus, we must have

$$\left[X^\alpha, X^\beta\right]_t = \int_0^t \langle P_\alpha, P_\beta\rangle(X_s)\,\mathrm{d}s\ .$$

Using (4.14), the Kunita-Watanabe identity and (4.15), we further deduce that

$$\begin{aligned}
\mathrm{d}\left[W^i, W^j\right]_t &= \sum_{\alpha,\beta=1}^N \langle \eta_\alpha, U_t e_i\rangle\langle \eta_\beta, U_t e_j\rangle \mathrm{d}\left[N^\alpha, N^\beta\right]_t \\
&= \sum_{\alpha,\beta=1}^N \langle \eta_\alpha, U_t e_i\rangle\langle \eta_\beta, U_t e_j\rangle\langle P_\alpha, P_\beta\rangle(X_t)\,\mathrm{d}t
\end{aligned} \tag{4.17}$$

for $1 \le i, j \le d$. If we now set $F_i = U_t e_i$, one can write

$$(P_\alpha)_{X_t} = \sum_{i=1}^d \langle F_i, \eta_\alpha\rangle F_i$$

because $\{F_i\}_{1\le i\le d}$ is an orthonormal basis of $T_{X_t}M$. Using this expression, we conclude

$$\begin{aligned}
\sum_{\alpha,\beta=1}^N \langle \eta_\alpha, F_i\rangle\langle \eta_\beta, F_j\rangle\langle P_\alpha, P_\beta\rangle(X_t) &= \sum_{\alpha,\beta=1}^N \sum_{k,l=1}^d \langle \eta_\alpha, F_i\rangle\langle \eta_\beta, F_j\rangle\langle F_k, \eta_\alpha\rangle\langle F_l, \eta_\beta\rangle\langle F_k, F_l\rangle \\
&= \sum_{k,l=1}^d \langle F_i, F_k\rangle\langle F_j, F_l\rangle\delta_{kl} = \delta_{ij}\ ,
\end{aligned}$$

by Parseval's identity. From (4.17) we then get

$$\left[W^i, W^j\right]_t = t\delta_{ij}\ .$$

By Lévy's characterisation and since $W_0 = 0$ due to the definition of the anti-development, it follows that W is indeed a standard Brownian motion on $\mathbb{R}^d$. □

Remark 4.8 *In Hsu* [9, Chapter 2.4] *Identity* (4.13) *is in fact used as the definition of the b-quadratic variation of X. However, this definition is equivalent to the one we took from Emery* [3, Chapter 3].

Remark 4.9 *Having established the characterisation of Brownian motion which uses stochastic differential equations, one can prove the existence of Brownian motion on a Riemannian manifold M up to explosion. We start with a standard Brownian motion W on $\mathbb{R}^d$, which does exist, and then solve the stochastic differential equation* (3.4)*. By Theorem* 3.3 *we are guaranteed the existence of a solution U on $\mathcal{O}(M)$ up to explosion. At the end, we project U onto M to obtain a Brownian motion X on M.*

We can now give our first example of a Brownian motion on a non-trivial manifold.

Example 4.10 Let $V = (V_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$. We claim that $X = (X_t)_{t\geq 0}$ given by

$$X_t = \mathrm{e}^{\mathrm{i}\, V_t}$$

is a Brownian motion on S^1. By Theorem 4.7, it suffices to show that the anti-development W of X is a standard Brownian motion on $\mathbb{R}$.

Let U_0 be an $\mathcal{F}_0$-measurable random variable on $\mathcal{O}(S^1)$ with $\pi(U_0) = X_0$. Since we have

$$\mathcal{O}(S^1) = S^1 \times \mathrm{O}(1) = S^1 \times \{\pm 1\}\ ,$$

the random variable U_0 must be of the form $U_0 = (X_0, B)$ for a random variable B taking values in $\{\pm 1\}$. We observe that (X_t, B) is the only continuous semimartingale on $\mathcal{O}(S^1)$ which lies above X and starts from U_0. As we are guaranteed the existence of a horizontal lift $U = (U_t)_{t\geq 0}$ starting from U_0, it follows that

$$U_t = (X_t, B)\ . \tag{4.18}$$

In general, one can think of an element $(x, \pm 1) \in \mathcal{O}(S^1)$ as the linear isometry from $\mathbb{R}$ to $T_x S^1$ which sends 1 to $\pm\,\mathrm{i}\, x$. Furthermore, the structure group of $\mathcal{O}(S^1)$ is $G = \{\pm 1\}$. Since its associated Lie algebra is $\mathfrak{g} = \{0\}$, any connection form on $\mathcal{O}(S^1)$ must be identically zero. This implies that $H\mathcal{O}(S^1) = T\mathcal{O}(S^1)$, i.e. all tangent vectors are horizontal. We deduce that the horizontal vector field H_1 which corresponds to the basis element 1 of $\mathbb{R}$ is given by

$$H_1\,(x, \varepsilon) = \begin{cases} (\mathrm{i}\, x, 0) & \text{if } \varepsilon = 1 \\ (-\,\mathrm{i}\, x, 0) & \text{if } \varepsilon = -1 \end{cases}$$

for $(x, \varepsilon) \in S^1 \times \{\pm 1\} = \mathcal{O}(S^1)$. In particular, we obtain $H_1(U_t) = (\mathrm{i}\, B X_t, 0) = (\mathrm{i}\, B\, \mathrm{e}^{\mathrm{i}\, V_t}, 0)$. Moreover, from (4.18) and the chain rule for Stratonovich differentials, we also have

$$\partial U_t = (\partial X_t, 0) = (\mathrm{i}\, \mathrm{e}^{\mathrm{i}\, V_t}\ \partial V_t, 0)\ .$$

Hence, $\partial U_t = H_1(U_t)\, \partial W_t$ yields

$$\mathrm{i}\, \mathrm{e}^{\mathrm{i}\, V_t}\ \partial V_t = \mathrm{i}\, B\, \mathrm{e}^{\mathrm{i}\, V_t}\ \partial W_t$$

which is equivalent to $\partial V_t = B\, \partial W_t$. Since we require $W_0 = 0$ for the anti-development, it follows that

$$W_t = \frac{V_t - V_0}{B}\ .$$

Hereby, we can divide by B as it takes values in $\{\pm 1\}$ only. Furthermore, since U_0 is $\mathcal{F}_0$-measurable and as $V = (V_t)_{t\geq 0}$ is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geq 0}$, the process V must be independent of B. Thus, W is indeed a standard Brownian motion on $\mathbb{R}$ since V is a Brownian motion on $\mathbb{R}$. $\square$

At the end of this section, we want to show that Proposition 1.5 is really just a special case of Theorem 4.7. It suffices to prove the following, without making use of Theorem 4.7.

Proposition 4.11 *A semimartingale $X = (X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is a Brownian motion on $\mathbb{R}^d$ if and only if its anti-development W is a standard Brownian motion on $\mathbb{R}^d$.*

Proof. We certainly have

$$\mathcal{O}(\mathbb{R}^d) = \mathbb{R}^d \times \mathrm{O}(d)\ .$$

Furthermore, one can show that the decomposition

$$T_u\mathcal{O}(\mathbb{R}^d) = T_u\mathbb{R}^d \otimes T_u\, \mathrm{O}(d)$$

which assigns $T_u\mathbb{R}^d \cong \mathbb{R}^d$ as the horizontal subspace to $u \in \mathcal{O}(\mathbb{R}^d)$ yields the connection on $\mathcal{O}(\mathbb{R}^d)$ which corresponds to the Levi-Civita connection on $\mathbb{R}^d$.

Similar to the previous example, any $\mathcal{F}_0$-measurable random variable U_0 on $\mathcal{O}(\mathbb{R}^d)$ with $\pi(U_0) = X_0$ is of the form $U_0 = (X_0, B)$ for a random variable B taking values in $\mathrm{O}(d)$. We claim that the horizontal lift U of X to $\mathcal{O}(\mathbb{R}^d)$ starting from U_0 is

$$U_t = (X_t, B) \ .$$

We certainly have $\pi(U) = X$. Therefore, it suffices to find a semimartingale W on $\mathbb{R}^d$ which satisfies (3.4). We note that with respect to the specified connection, the horizontal vector fields $H_1, H_2, \ldots, H_d$ are given by

$$H_i(u) = (Ae_i, 0) \quad \text{for } u = (x, A) \ ,$$

where $\{e_i\}_{1 \le i \le d}$ is the standard basis of $\mathbb{R}^d$. Thus, due to $\partial U_t = (\partial X_t, 0)$ the stochastic differential equation $\partial U_t = \sum_{i=1}^d H_i(U_t)\,\partial W_t^i$ reads

$$(\partial X_t, 0) = \sum_{i=1}^d (Be_i, 0)\,\partial W_t^i \ .$$

It follows that

$$\partial X_t = \sum_{i=1}^d Be_i\,\partial W_t^i \ .$$

Hence, the anti-development W of X is given by

$$W_t = B^{-1}(X_t - X_0) \ . \tag{4.19}$$

Note that B^{-1} is well-defined because B is a random variable on $\mathrm{O}(d)$. It particularly follows that $U_t = (X_t, B)$ is indeed the horizontal lift of X starting from (X_0, B). Moreover, if $W = (W_t)_{t \ge 0}$ is a standard Brownian motion on $\mathbb{R}^d$ in the filtration $(\mathcal{F}_t)_{t \ge 0}$ then B is independent of W. Thus, as B takes values in $\mathrm{O}(d)$ only, Equation (4.19) implies that X is a Brownian motion on $\mathbb{R}^d$. On the other hand, if $X = (X_t)_{t \ge 0}$ is a Brownian motion on $\mathbb{R}^d$ in the filtration $(\mathcal{F}_t)_{t \ge 0}$ then B is independent of X and it follows that W is a standard Brownian motion on $\mathbb{R}^d$. $\square$

4.2 Discrete approximation of Brownian motion

In this section, we present a discrete approximation of Brownian motion on a Riemannian manifold. This also gives us a better idea of how one could think about Brownian motion. For the most part, we follow Feres [4, Chapter 8].

Let M be a d-dimensional Riemannian manifold. In this section only, we shall additionally assume that M is compact. On the one hand, this guarantees the completeness of every horizontal vector field on $\mathcal{O}(M)$ and on the other hand, this assumption ensures that stochastic processes on M and on $\mathcal{O}(M)$ do not explode. To find a discrete approximation of Brownian motion on M, we first use a standard Brownian motion W on $\mathbb{R}^d$ and the horizontal vector fields H_i to construct a discrete approximation of its horizontal lift on $\mathcal{O}(M)$. Afterwards, we project this approximation onto M.

For a horizontal vector field H on $\mathcal{O}(M)$ let $\left(\Phi_t^H\right)$ denote the flow of H, i.e. $\Phi_t^H(u)$ is the unique integral curve of H through $u \in \mathcal{O}(M)$. By our assumptions, this flow is defined for all times t. Let $x \in M$ be fixed and let U_0 be a random variable taking values in $\mathcal{O}(M)_x$. For some fixed time $T > 0$ and for each $n \in \mathbb{N}$, we define a curve $U^{(n)} = \left(U_t^{(n)}\right)_{T \ge t \ge 0}$ on $\mathcal{O}(M)$ in the following way. Set $U_0^{(n)} = U_0$. We then use the horizontal vector field

$$H_t^n = \sum_{i=1}^d \left(W_t^i - W_{k2^{-n}T}^i\right) H_i$$

for $t \in [k2^{-n}T, (k+1)2^{-n}T]$ and $0 \leq k \leq 2^n - 1$ to inductively construct

$$U_t^{(n)} = \Phi_1^{H_t^n}\left(U_{k2^{-n}T}^{(n)}\right) .$$

Note that this is consistent for $t = k2^{-n}T$ as H_t^n vanishes at $t = k2^{-n}T$.

We are mainly interested in the projection $X^{(n)} = \left(X_t^{(n)}\right)_{T \geq t \geq 0}$ given by $X^{(n)} = \pi\left(U^{(n)}\right)$. Using the correspondence between the connection on $\mathcal{O}(M)$ and the Levi-Civita connection on M, one can show that for $t \in [k2^{-n}T, (k+1)2^{-n}T]$ we reach $X_t^{(n)}$ by walking a parameter distance 1 along the geodesic through $X_{k2^{-n}T}^{(n)}$ with tangent vector

$$\sum_{i=1}^{d}\left(W_t^i - W_{k2^{-n}T}^i\right) U_{k2^{-n}T}^{(n)} e_i$$

at $X_{k2^{-n}T}^{(n)}$.

The following theorem says that the sequence of processes $\left(U^{(n)}\right)_{n \in \mathbb{N}}$ has the desired limit, i.e. it approximates the horizontal lift.

Theorem 4.12 *Let M be a Riemannian manifold, let $x \in M$ be fixed and let U_0 be a random variable on $\mathcal{O}(M)_x$. Then there exists a stochastic process $U = (U_t)_{t \geq 0}$ on $\mathcal{O}(M)$ starting from U_0 such that for every $T > 0$ and every smooth function $f \in C^\infty(\mathcal{O}(M))$*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \left| f\left(U_t^{(n)}\right) - f\left(U_t\right)\right| = 0 \qquad a.s. ,$$

where $U^{(n)} = \left(U_t^{(n)}\right)_{T \geq t \geq 0}$ are the stochastic processes defined above. Moreover, for every smooth function $g \in C^\infty\left([0,\infty) \times \mathcal{O}(M)\right)$ the stochastic process $N = (N_t)_{t \geq 0}$ given by

$$N_t = g\left(t, U_t\right) - g\left(0, U_0\right) - \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_{\mathcal{O}(M)}\right) g\left(s, U_s\right) \, \mathrm{d}s$$

is a local martingale on $\mathbb{R}$.

A proof of a more general version of this theorem is included in Feres [4, Chapter 8].

Let $X = (X_t)_{t \geq 0}$ be the stochastic process on M given by $X = \pi\left(U\right)$. For an arbitrary $f \in C^\infty(M)$, we consider the function $g \in C^\infty\left([0,\infty) \times \mathcal{O}(M)\right)$ which is defined by $g(t,u) = (f \circ \pi)(u)$. Applying the second part of Theorem 4.12 to g shows that

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_{\mathcal{O}(M)}(f \circ \pi)\left(U_s\right) \, \mathrm{d}s$$

is a local martingale. Using (4.10), we deduce that

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \Delta_M f(X_s) \, \mathrm{d}s$$

is a local martingale on $\mathbb{R}$. Hence, X is a Brownian motion on the Riemannian manifold M and $\left(X^{(n)}\right)_{n \in \mathbb{N}}$ is indeed a sequence approximating Brownian motion on M.

The constructions we presented in the last two sections are generally known as 'rolling without slipping'. They also provide an intuitive picture of how to obtain Brownian motion on a Riemannian manifold, see e.g. Rogers, Williams [16, V. 33]. We think of the d-dimensional Riemannian manifold

M as embedded into $\mathbb{R}^N$ for some $N \in \mathbb{N}$. To construct Brownian motion on M starting at x we first construct a standard Brownian motion on $L \cong \mathbb{R}^d$ and place L tangential to M at x such that the origin of L coincides with x. We then roll L without slipping on M such that at time t the point $W_t \in L$ is in contact with M. The corresponding points $(X_t)_{t\geq 0}$ on M form a Brownian motion X on M. Moreover, if we mark coordinate axes on L, then at each point X_t these axes provide a choice of an orthonormal basis for the tangent space $T_{X_t}M$. This corresponds to the horizontal lift U of X to the orthonormal frame bundle $\mathcal{O}(M)$.

Remark 4.13 *It is possible to define the concept of a random walk on a Riemannian manifold. As in the Euclidean case, one can then construct Brownian motion as the limit of a sequence of random walks. An exposition of this idea can be found in Jørgensen* [10].

4.3 Characterising Brownian motion via the heat equation

As before, let M be a d-dimensional Riemannian manifold. Moreover, let $X = (X_t)_{e>t\geq 0}$ be a Brownian motion on M starting at $X_0 = x$ and let $\mathbb{P}_x$ denote its law. In this section, we aim to find the transition density function of X, i.e. we want to find a non-negative function $p_M(t,x,y)$ defined on $(0,\infty)\times M \times M$ satisfying

$$\mathbb{P}_x(X_t \in C, t < e) = \int_C p_M(t,x,y)\,\mathrm{d}y$$

for $t > 0$ and any Borel subset C of M, cf. Proposition 4.18. Note that the integral is understood to be with respect to the Riemannian volume measure.

To find such a function, we follow the approach presented in Hsu [9, Chapter 4.1]. In the following, let $\mathscr{L}_M$ denote the heat operator on M. It is applied to functions $g \in C^{1,2}\left((0,\infty)\times M\right)$ and is defined by

$$\mathscr{L}_M = \frac{\partial}{\partial t} - \frac{1}{2}\Delta_M \ .$$

Thereby, $\frac{\partial}{\partial t}$ is applied to the time coordinate of g and Δ_M to its spatial coordinates.

Theorem 4.14 *Let M be a Riemannian manifold and let $D \subset M$ be a relatively compact domain with smooth boundary ∂D. There exists a unique continuous function $p_D(t,x,y)$ defined on $(0,\infty)\times \overline{D}\times\overline{D}$ satisfying the following conditions.*

(i) *$p_D(t,x,y)$ is strictly positive and infinitely differentiable on $(0,\infty)\times D\times D$.*

(ii) *For every fixed $y \in D$, the function $q(t,x) = p_D(t,x,y)$ on $(0,\infty)\times D$ is a solution of the heat equation, i.e.*
$$\mathscr{L}_M q(t,x) = 0 \ \ \textit{for all} \ \ (t,x) \in (0,\infty)\times D \ .$$

(iii) *For every bounded continuous function f on D and every $y \in D$ it holds true that*
$$\lim_{t\downarrow 0}\int_D p_D(t,x,y)f(x)\,\mathrm{d}x = f(y) \ .$$

(iv) *For every $y \in \partial D$ we have $p_D(t,x,y) = 0$ for all $(t,x)\in(0,\infty)\times D$.*

(v) *The function is symmetric in its last two arguments, i.e. $p_D(t,x,y) = p_D(t,y,x)$.*

(vi) *For any $x, y \in D$ and $t, s > 0$ the Chapman-Kolmogorov equation*
$$p_D(t+s,x,y) = \int_D p_D(t,x,z)p_D(s,z,y)\,\mathrm{d}z$$
holds true.

(vii) *For every fixed pair* $(t,x) \in (0,\infty) \times D$ *we have* $\int_D p_D(t,x,y)\,\mathrm{d}y \le 1$. *This inequality is strict if* $M \setminus \overline{D} \neq \emptyset$.

We omit the proof here as it does not use any techniques which are of further interest to this essay. For more details, we refer to Grigor'yan [6, Chapter 7 and 8].

Since we would like to consider $p_D(t,x,y)$ as a function on $(0,\infty) \times M \times M$, we agree to set $p_D(t,x,y) = 0$ if $x \in M \setminus \overline{D}$ or $y \in M \setminus \overline{D}$. In particular, with that convention, we have

$$\int_D p_D(t,x,y) f(y)\,\mathrm{d}y = \int_M p_D(t,x,y) f(y)\,\mathrm{d}y$$

for any continuous function f on M.

The next lemma, which is needed later, is a consequence of the preceding theorem.

Lemma 4.15 *Let* $D \subset M$ *be a relatively compact domain with smooth boundary and let* f *be a continuous function on* M. *Then*

$$u(t,x) = \int_M p_D(t,x,y) f(y)\,\mathrm{d}y \tag{4.20}$$

defined on $(0,\infty) \times \overline{D}$ *satisfies the heat equation*

$$\mathscr{L}_M u(t,x) = 0 \quad \text{for all} \quad (t,x) \in (0,\infty) \times D\ ,$$

the boundary condition $u(t,x) = 0$ *for* $t > 0$ *and* $x \in \partial D$ *as well as the initial condition*

$$\lim_{t \downarrow 0} u(t,x) = f(x) \quad \text{for all} \quad x \in D\ .$$

Sketch of proof. The function defined in (4.20) has the claimed boundary condition by Theorem 4.14 (iv) and (v) as well as the claimed initial condition by Theorem 4.14 (iii) and (v).
To prove that u solves the heat equation, one first has to argue that one can interchange the heat operator $\mathscr{L}_M$ and the integral to obtain

$$\mathscr{L}_M u(t,x) = \int_M \mathscr{L}_M p_D(t,x,y) f(y)\,\mathrm{d}y\ .$$

Since it also holds true that

$$\int_M \mathscr{L}_M p_D(t,x,y) f(y)\,\mathrm{d}y = \int_D \mathscr{L}_M p_D(t,x,y) f(y)\,\mathrm{d}y$$

the desired result then follows by Theorem 4.14 (ii). □

In fact, (4.20) is even the unique solution to the initial-boundary value problem given in Lemma 4.15, cf. Hsu [9, Chapter 4.1].

The following lemma, see e.g. Stroock, Varadhan [17, Theorem 4.2.1], is also needed to prove the next proposition.

Lemma 4.16 *Let* $X = (X_t)_{e>t\ge 0}$ *be a Brownian motion on a Riemannian manifold* M *and let* $g \in C^\infty([0,\infty) \times M)$. *Then*

$$g(t,X_t) - g(0,X_0) - \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_M\right) g(s,X_s)\,\mathrm{d}s$$

is a local martingale on $\mathbb{R}$.

Proof. Let $N = (N_t)_{e>t\geq 0}$ be the stochastic process given by

$$N_t = g(0, X_t) - g(0, X_0) - \frac{1}{2}\int_0^t \Delta_M g(0, X_r)\,\mathrm{d}r\ .$$

Since X is a Brownian motion on M and as $g(0,\cdot) \in C^\infty(M)$, it follows that N is a local martingale on $\mathbb{R}$. Moreover, we deduce that

$$\begin{aligned}
g(t, X_t) - g(0, X_0) &= g(t, X_t) - g(0, X_t) + g(0, X_t) - g(0, X_0) \\
&= \int_0^t \left(\frac{\partial g}{\partial s}\right)(s, X_t)\,\mathrm{d}s + \frac{1}{2}\int_0^t \Delta_M g(0, X_r)\,\mathrm{d}r + N_t \\
&= \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_M\right) g(s, X_s)\,\mathrm{d}s + N_t \\
&\quad + \int_0^t \left(\frac{\partial g}{\partial s}\right)(s, X_t)\,\mathrm{d}s - \int_0^t \left(\frac{\partial g}{\partial s}\right)(s, X_s)\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \Delta_M g(0, X_r)\,\mathrm{d}r - \frac{1}{2}\int_0^t \Delta_M g(r, X_r)\,\mathrm{d}r\ .
\end{aligned}$$

For each $s \geq 0$, we set

$$\widetilde{N}_t(s) = \left(\frac{\partial g}{\partial s}\right)(s, X_t) - \left(\frac{\partial g}{\partial s}\right)(s, X_0) - \frac{1}{2}\int_0^t \Delta_M \left(\frac{\partial g}{\partial s}\right)(s, X_r)\,\mathrm{d}r \tag{4.21}$$

and observe that $\widetilde{N}(s)$ is a local martingale because $(\partial g/\partial s)(s,\cdot) \in C^\infty(M)$. We also note that

$$\widetilde{N}_t(s) - \widetilde{N}_s(s) = \left(\frac{\partial g}{\partial s}\right)(s, X_t) - \left(\frac{\partial g}{\partial s}\right)(s, X_s) - \frac{1}{2}\int_s^t \Delta_M \left(\frac{\partial g}{\partial s}\right)(s, X_r)\,\mathrm{d}r\ .$$

Furthermore, we can write

$$\Delta_M g(0, X_r) - \Delta_M g(r, X_r) = -\int_0^r \left(\frac{\partial}{\partial s}\right) \Delta_M g(s, X_r)\,\mathrm{d}s\ .$$

Putting these expressions into the previously established identity yields

$$\begin{aligned}
g(t, X_t) - g(0, X_0) = &\int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_M\right) g(s, X_s)\,\mathrm{d}s + N_t \\
&+ \frac{1}{2}\int_0^t \int_s^t \Delta_M \left(\frac{\partial g}{\partial s}\right)(s, X_r)\,\mathrm{d}r\,\mathrm{d}s + \int_0^t \left(\widetilde{N}_t(s) - \widetilde{N}_s(s)\right)\,\mathrm{d}s \\
&- \frac{1}{2}\int_0^t \int_0^r \left(\frac{\partial}{\partial s}\right) \Delta_M g(s, X_r)\,\mathrm{d}s\,\mathrm{d}r\ .
\end{aligned}$$

The two double integrals cancel each other because $\Delta_M(\partial g/\partial s) = (\partial/\partial s)\Delta_M g$ and in both cases the region of integration is $0 \leq s \leq r \leq t$. To establish the desired conclusion, it remains to prove that

$$\int_0^t \left(\widetilde{N}_t(s) - \widetilde{N}_s(s)\right)\,\mathrm{d}s \tag{4.22}$$

is also local martingale. By using the Riemannian metric h on M, we can define the distance $d(x, y)$ between two point $x, y \in M$. If

$$\sup_{e>t\geq 0} d(X_0, X_t)$$

is a.s. bounded, let $\left(\widetilde{T}_n\right)_{n\in\mathbb{N}}$ be a sequence of stopping times increasing to the explosion time and otherwise, let

$$\widetilde{T}_n = \inf\{t \geq 0\colon d(X_0, X_t) > n\} \ .$$

We then define $T_n = n \wedge \widetilde{T}_n$ and note that $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times increasing to the explosion time. Moreover, we claim that for every $s \in [0, e)$ the sequence $(T_n)_{n\in\mathbb{N}}$ reduces the local martingale $\widetilde{N}(s)$. Indeed, by definition there exists a sequence of stopping times $(S_m)_{m\in\mathbb{N}}$ increasing to the explosion time e such that $\widetilde{N}(s)^{S_m}$ is a martingale for each $m \in \mathbb{N}$. This implies that

$$\left(\widetilde{N}(s)^{S_m}\right)^{T_n}$$

is a martingale for each $n, m \in \mathbb{N}$. Furthermore, by examining (4.21) we observe that $\widetilde{N}(s)^{T_n}$ is bounded. Thus, by the dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{E}\left[\widetilde{N}_{t_1\wedge T_n}(s)\middle|\mathcal{F}_{t_2}\right] &= \mathbb{E}\left[\lim_{m\to\infty}\widetilde{N}_{t_1\wedge T_n\wedge S_m}(s)\middle|\mathcal{F}_{t_2}\right] \\
&= \lim_{m\to\infty}\mathbb{E}\left[\widetilde{N}_{t_1\wedge T_n\wedge S_m}(s)\middle|\mathcal{F}_{t_2}\right] \\
&= \lim_{m\to\infty}\widetilde{N}_{t_2\wedge T_n\wedge S_m}(s) = \widetilde{N}_{t_2\wedge T_n}(s)
\end{aligned}$$

for $e > t_1 > t_2 \geq 0$. Due to $T_n \leq n$ and from (4.21), we also obtain that

$$\sup_{0\leq s,t\leq T_n}\widetilde{N}_t(s) < \infty$$

for any $n \in \mathbb{N}$. Hence, we can use Fubini's Theorem to argue that for $e > t_1 > t_2 \geq 0$ and $n \in \mathbb{N}$

$$\mathbb{E}\left[\int_0^{t_1\wedge T_n}\left(\widetilde{N}_{t_1\wedge T_n}(s) - \widetilde{N}_s(s)\right)\mathrm{d}s\middle|\mathcal{F}_{t_2}\right] = \int_0^{t_1\wedge T_n}\mathbb{E}\left[\widetilde{N}_{t_1\wedge T_n}(s) - \widetilde{N}_{s\wedge T_n}(s)\middle|\mathcal{F}_{t_2}\right]\mathrm{d}s \ .$$

Since

$$\mathbb{E}\left[\widetilde{N}_{s\wedge T_n}(s)\middle|\mathcal{F}_{t_2}\right] = \begin{cases}\widetilde{N}_{t_2\wedge T_n}(s) & \text{if } s > t_2\\ \widetilde{N}_{s\wedge T_n}(s) & \text{if } s \leq t_2\end{cases}$$

it follows that

$$\begin{aligned}
\int_0^{t_1\wedge T_n}\mathbb{E}\left[\widetilde{N}_{t_1\wedge T_n}(s) - \widetilde{N}_{s\wedge T_n}(s)\middle|\mathcal{F}_{t_2}\right]\mathrm{d}s &= \int_0^{t_1\wedge T_n}\widetilde{N}_{t_2\wedge T_n}(s)\,\mathrm{d}s - \int_0^{t_2\wedge T_n}\widetilde{N}_{s\wedge T_n}(s)\,\mathrm{d}s \\
&\quad - \int_{t_2\wedge T_n}^{t_1\wedge T_n}\widetilde{N}_{t_2\wedge T_n}(s)\,\mathrm{d}s \\
&= \int_0^{t_2\wedge T_n}\left(\widetilde{N}_{t_2\wedge T_n}(s) - \widetilde{N}_s(s)\right)\mathrm{d}s \ .
\end{aligned}$$

Thus, $(T_n)_{n\in\mathbb{N}}$ reduces the expression given in (4.22) which implies that the latter is indeed a local martingale. □

For an open subset $U \subset M$, let

$$\tau_U = \inf\{t < e\colon X_t \notin U\}$$

be the first exit time from U, where we agree to set $\tau_U = e$ if $\{t < e\colon X_t \notin U\}$ is empty. We note that the random time τ_U is a stopping time because $M \setminus U$ is closed.

The following proposition says that $p_D(t, x, y)$ is the transition density function of Brownian motion on M which starts at $x \in D$ and gets killed at the boundary of D. For the proof, we use the ideas given in Hsu [9, Chapter 4.1] and add in the details, such as the previous lemma.

Proposition 4.17 *Let M be a Riemannian manifold and let $D \subset M$ be a relatively compact domain with smooth boundary. If X is a Brownian motion on M starting at $x \in D$ then*

$$\mathbb{P}_x(X_t \in C, t < \tau_D) = \int_C p_D(t,x,y)\,\mathrm{d}y$$

for $t > 0$ and any Borel subset C of M.

Proof. Let f be a continuous function on M and consider the function $u(s,y)$ on $(0,\infty)\times M$ defined by

$$u(s,y) = \int_M p_D(s,y,z) f(z)\,\mathrm{d}z\ .$$

For a fixed $t \in (0,\infty)$, let $v(s,y)$ be the function on $[0,t)\times M$ given by

$$v(s,y) = u(t-s,y)\ .$$

The restriction $s < t$ ensures that $u(t-s,y)$ is well-defined. We note that v is continuous on $[0,t)\times M$ and smooth on $[0,t)\times D$. Therefore, by Lemma 4.16, the process $N = (N_s)_{t\wedge\tau_D > s \geq 0}$ given by

$$N_s = v(s,X_s) - v(0,X_0) - \int_0^s \left(\frac{\partial}{\partial r} + \frac{1}{2}\Delta_M\right) v(r,X_r)\,\mathrm{d}r$$

is a local martingale on $\mathbb{R}$. Furthermore, we also have

$$-\left(\frac{\partial}{\partial r} + \frac{1}{2}\Delta_M\right) v(r,X_r) = \mathscr{L}_M u(t-r,X_r)\ .$$

By Lemma 4.15, we know that $\mathscr{L}_M u(t-r,X_r) = 0$ as long as $X_r \in D$. Since we assume $s \in [0, t\wedge\tau_D)$ it follows that

$$N_s = v(s,X_s) - v(0,X_0) = u(t-s,X_s) - u(t,x)\ .$$

As f is continuous, it is bounded on the compact set $\overline{D}$. Thus, since $p_D(s,y,z) = 0$ if $y \in M\setminus D$ or $z \in M \setminus D$ and due to Theorem 4.14 (vii) the functions u and v are bounded. In particular, this implies that N is a true martingale. As $N_0 = 0$, we further deduce that $\mathbb{E}_x[N_s] = 0$, i.e.

$$\mathbb{E}_x u(t-s,X_s) = u(t,x) \tag{4.23}$$

for $0 \leq s < t\wedge\tau_D$. For a fixed $t_0 < t$, we note that

$$u(t-s,X_s) \to u(t-t_0\wedge\tau_D, X_{t_0\wedge\tau_D}) \quad \text{as} \quad s \uparrow t_0\wedge\tau_D$$

by the continuity of u on $(0,\infty)\times\overline{D}$ and since X has continuous sample paths. Using the dominated convergence theorem and (4.23), it follows that

$$\mathbb{E}_x u(t-t_0\wedge\tau_D, X_{t_0\wedge\tau_D}) = u(t,x)\ .$$

On the other hand, since $u(s,y) = 0$ if $y \in \partial D$ and as $X_{\tau_D} \in \partial D$, we also have

$$u(t-t_0\wedge\tau_D, X_{t_0\wedge\tau_D}) = \begin{cases} u(t-t_0, X_{t_0}) & \text{if } \tau_D > t_0 \\ 0 & \text{if } \tau_D \leq t_0\ , \end{cases}$$

which implies

$$u(t,x) = \mathbb{E}_x u(t-t_0\wedge\tau_D, X_{t_0\wedge\tau_D}) = \mathbb{E}_x\left[u(t-t_0,X_{t_0})\mathbb{1}_{\{t_0<\tau_D\}}\right]\ . \tag{4.24}$$

Let us now consider a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,t)$ strictly increasing to t. From Lemma 4.15 we know that for $y \in D$ the function $u(s,y)$ satisfies the initial condition

$$\lim_{s\downarrow 0} u(s,y) = f(y)\ .$$

Using the fact that D is relatively compact, one then argues that $u(t-t_n, X_{t_n})\mathbb{1}_{\{t_n<\tau_D\}}$ tends to $f(X_t)\mathbb{1}_{\{t<\tau_D\}}$ pointwise as $n\to\infty$. Furthermore, due to the boundedness of u, we can apply the dominated convergence theorem to deduce that

$$\mathbb{E}_x\left[u(t-t_n, X_{t_n})\mathbb{1}_{\{t_n<\tau_D\}}\right] \to \mathbb{E}_x\left[f(X_t)\mathbb{1}_{\{t<\tau_D\}}\right] \qquad \text{as } n\to\infty\ .$$

Since $t_0 < t$ was arbitrary in (4.24), it follows that

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{\{t<\tau_D\}}\right] = u(t,x) = \int_M p_D(t,x,z)f(z)\,\mathrm{d}z\ . \tag{4.25}$$

Finally, let $C\subset M$ be a Borel subset. One can find an increasing sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ on M such that for almost every $x\in M$ the values $f_n(x)$ tend to $\mathbb{1}_C(x)$ as $n\to\infty$. Using (4.25) and the monotone convergence theorem, we obtain

$$\mathbb{P}_x(X_t\in C, t<\tau_D) = \mathbb{E}_x\left[\mathbb{1}_C(X_t)\mathbb{1}_{\{t<\tau_D\}}\right] = \int_M p_D(t,x,z)\mathbb{1}_C(z)\,\mathrm{d}z = \int_C p_D(t,x,z)\,\mathrm{d}z\ ,$$

as claimed. □

To extend this result to M, we consider a sequence $(D_n)_{n\in\mathbb{N}}$ of relatively compact domains with smooth boundary and such that

(i) $\overline{D_n}\subset D_{n+1}$ for all $n\in\mathbb{N}$ and

(ii) $\bigcup_{n\in\mathbb{N}} D_n = M$.

One can show that such a sequence always exists. We refer to $(D_n)_{n\in\mathbb{N}}$ as an exhaustion sequence of M.

Let X be a Brownian motion starting at $x\in D_n$. From Proposition 4.17, we deduce that for any Borel set $C\subset M$ and for $t>0$

$$\int_C \left(p_{D_{n+1}}(t,x,y) - p_{D_n}(t,x,y)\right)\,\mathrm{d}y = \mathbb{P}_x(X_t\in C, \tau_{D_n}\le t<\tau_{D_{n+1}}) \ge 0\ .$$

Since we also have $p_{D_n}(t,x,y)=0$ for $x\in M\setminus D_n$ the non-negativity of $p_{D_{n+1}}(t,x,y)$ implies that

$$p_{D_{n+1}}(t,x,y) \ge p_{D_n}(t,x,y)$$

holds true everywhere. Thus, we can define

$$p_M(t,x,y) = \lim_{n\to\infty} p_{D_n}(t,x,y)\ . \tag{4.26}$$

The next proposition shows that $p_M(t,x,y)$ is the transition density function of Brownian motion on M starting at x.

Proposition 4.18 *Let M be a Riemannian manifold. If $X=(X_t)_{e>t\ge 0}$ is a Brownian motion on M starting at $x\in M$ then*

$$\mathbb{P}_x(X_t\in C, t<e) = \int_C p_M(t,x,y)\,\mathrm{d}y$$

for $t>0$ and any Borel subset C of M.

Proof. Let $C\subset M$ be an arbitrary Borel set and let $t>0$. Due to the properties of the exhaustion sequence $(D_n)_{n\in\mathbb{N}}$, there exists some $N\in\mathbb{N}$ such that $x\in D_n$ for all $n\ge N$. By Proposition 4.17, it then holds true that

$$\mathbb{P}_x(X_t\in C, t<\tau_{D_n}) = \int_C p_{D_n}(t,x,y)\,\mathrm{d}y \tag{4.27}$$

for all $n \geq N$. Since $p_{D_n}(t,x,y) \uparrow p_M(t,x,y)$ as $n \to \infty$ the monotone convergence theorem implies

$$\lim_{n\to\infty} \int_C p_{D_n}(t,x,y)\,\mathrm{d}y = \int_C p_M(t,x,y)\,\mathrm{d}y \ .$$

Moreover, we note that $\tau_{D_n} \uparrow e$ as $n \to \infty$. By again applying the monotone convergence theorem, we obtain

$$\lim_{n\to\infty} \mathbb{P}_x\left(X_t \in C, t < \tau_{D_n}\right) = \mathbb{P}_x\left(X_t \in C, t < e\right) \ .$$

Thus, the desired result follows by letting n tend to $+\infty$ in (4.27). □

From Proposition 4.18, we also deduce that $p_M(t,x,y)$ is independent of the choice of the exhaustion sequence $(D_n)_{n\in\mathbb{N}}$. In fact, $p_M(t,x,y)$ is the minimal heat kernel of M. This is a consequence of the next two theorems, both of which are taken from Hsu [9, Chapter 4.1]. In Theorem 4.19, one finds the main properties of $p_M(t,x,y)$, most of which follow from Theorem 4.14 and (4.26).

Theorem 4.19 *Let M be a Riemannian manifold. The function $p_M(t,x,y)$ satisfies the following conditions.*

(i) *$p_M(t,x,y)$ is strictly positive and infinitely differentiable on $(0,\infty) \times M \times M$.*

(ii) *For every fixed $y \in M$, the function $q(t,x) = p_M(t,x,y)$ on $(0,\infty) \times M$ is a solution of the heat equation, i.e.*

$$\mathscr{L}_M q(t,x) = 0 \quad \text{for all} \quad (t,x) \in (0,\infty) \times M \ .$$

(iii) *For every bounded continuous function f on M and every $y \in M$ it holds true that*

$$\lim_{t\downarrow 0} \int_M p_M(t,x,y) f(x)\,\mathrm{d}x = f(y) \ .$$

(iv) *The function is symmetric in its last two arguments, i.e. $p_M(t,x,y) = p_M(t,y,x)$.*

(v) *For any $x,y \in M$ and $t,s > 0$, the Chapman-Kolmogorov equation*

$$p_M(t+s,x,y) = \int_M p_M(t,x,z) p_M(s,z,y)\,\mathrm{d}z$$

holds true.

(vi) *For every fixed pair $(t,x) \in (0,\infty) \times M$ we have $\int_M p_M(t,x,y)\,\mathrm{d}y \leq 1$.*

The final theorem establishes a minimality property of $p_M(t,x,y)$. Even though we do not need this property in later parts of the essay, we state it here for the record.

Theorem 4.20 *If $p(t,x,y)$ is a function satisfying the conditions* (i) *to* (iii) *of the previous theorem then $p_M(t,x,y) \leq p(t,x,y)$ for all $t \in (0,\infty)$ and $x,y \in M$.*

A proof is given in Hsu [9, Chapter 4.1].

5 Recurrence and transience

Having established various characterisations of Brownian motion, we want to conclude the essay by analysing the recurrence and transience behaviour of Brownian motion on a Riemannian manifold. We start by giving the formal definitions of these two concepts and then check that Brownian motion is indeed either recurrent or transient. Afterwards, we find criterions which are equivalent to the transience of Brownian motion. We finish off by considering some Riemannian manifolds for which one can explicitly determine whether Brownian motion on them is recurrent or transient.

Throughout this chapter, we assume that the stochastic process $X = (X_t)_{e>t\geq 0}$ defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a Brownian motion on the Riemannian manifold M and in the filtration $(\mathcal{F}_t)_{t\geq 0}$. We denote its starting point by x and its law by $\mathbb{P}_x$.

5.1 Definitions and basic properties

The main ideas are taken from Hsu [9, Chapter 4.4]. However, I modified a few details to ensure that the arguments presented do actually work.

We observe that for a fixed $\omega \in \Omega$, one obtains a path $X(\omega)$ on M defined by $X(\omega) = (X_t(\omega))_{e(\omega)>t\geq 0}$.

Definition 5.1 *Let C be a Borel subset of M and let $\omega \in \Omega$ be fixed. We say that C is recurrent for the path $X(\omega)$ if there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}} \subset [0, e(\omega))$ such that*

$$\lim_{n\to\infty} t_n = e(\omega) \qquad \textit{and} \qquad X_{t_n}(\omega) \in C \quad \textit{for all } n \in \mathbb{N} \,.$$

Otherwise, i.e. if there exists some $T \in [0, e(\omega))$ such that $X_t(\omega) \notin C$ for all $t \geq T$, we say that C is transient for $X(\omega)$.

Definition 5.2 *A Borel subset C of M is called recurrent if for all $x \in M$ we have*

$$\mathbb{P}_x\left(\{X(\omega)\colon C \textit{ is recurrent for } X(\omega)\}\right) = 1 \,,$$

whereas it is called transient if for all $x \in M$

$$\mathbb{P}_x\left(\{X(\omega)\colon C \textit{ is transient for } X(\omega)\}\right) = 1 \,.$$

For a fixed $\omega \in \Omega$, Definition 5.1 ensures that a given Borel set $C \subset M$ is either recurrent or transient for $X(\omega)$. Thus,

$$\mathbb{P}_x\left(\{X(\omega)\colon C \text{ is recurrent for } X(\omega)\}\right) = 1 - \mathbb{P}_x\left(\{X(\omega)\colon C \text{ is transient for } X(\omega)\}\right) \tag{5.1}$$

and it follows that C cannot be both recurrent and transient.

Definition 5.3 *We say that Brownian motion on M is recurrent if every non-empty open subset of M is recurrent. Similarly, we call Brownian motion on M transient if every compact subset of M is transient.*

For the latter definition note that the empty set is always transient.

We would like to have the property that Brownian motion on M is neither recurrent and transient at the same time nor that it has none of these two features, cf. Corollary 5.5. However, this does not immediately follow from the last definition and we need to prove a series of lemmas to show that it is indeed the case.

In contrast to Hsu [9, Chapter 4.4], let $\mathscr{K}(M)$ be the set of all non-empty, compact and connected subsets of M which have a smooth boundary and a non-empty interior. The next proposition establishes that either all sets in $\mathscr{K}(M)$ are recurrent or all sets in $\mathscr{K}(M)$ are transient.

Proposition 5.4 *A subset $K \in \mathscr{K}(M)$ is either recurrent or transient. Moreover, if there exists some $L \in \mathscr{K}(M)$ which is recurrent then any $K \in \mathscr{K}(M)$ is recurrent.*

We claim this proposition implies that Brownian motion on M enjoys the desired property. In fact, the following corollary is not explicitly stated or proved in Hsu [9]. However, one can deduce it from Proposition 5.4.

Corollary 5.5 *Brownian motion on M is either recurrent or transient.*

Proof. Let C_1, C_2 be subsets of M satisfying $C_1 \subset C_2$. We observe that C_1 being recurrent implies the recurrence of C_2 whereas C_2 being transient implies the transience of C_1. The reason for this is that any path which hits C_1 has to meet C_2 at the same time, whereas any path which never comes back to C_2 can also never re-enter C_1.

Assume Brownian motion on M is not recurrent. By Definition 5.3, there exists a non-empty open subset $U \subset M$ which is not recurrent. Taking the closure of a geodesic ball with small radius and which lies inside U, one obtains a set $K_0 \in \mathscr{K}(M)$ satisfying $K_0 \subset U$. Hereby, we used the fact that a geodesic ball with small enough radius has a smooth boundary. Since U is not recurrent the set K_0 cannot be recurrent by our first observation. From Proposition 5.4, we deduce that every set contained in $\mathscr{K}(M)$ must be transient. Let L be a compact subset of M. One can find some $K_1 \in \mathscr{K}(M)$ with $L \subset K_1$. Since K_1 is transient, the set L must be transient as well. As L was an arbitrary compact subset, it follows that Brownian motion on M is transient. Hence, we established that Brownian motion on M not being recurrent implies that it is transient.

Conversely, suppose that Brownian motion on M is transient. By definition, this means that every compact subset of M is transient. In particular, any set contained in $\mathscr{K}(M)$ must be transient. Let U be a geodesic ball in M. Provided U has a small enough radius, we have $K = \overline{U} \in \mathscr{K}(M)$. As K is transient, U must be transient as well. By Equation (5.1), this implies that U cannot be recurrent. Since U is also a non-empty open subset of M, it follows that Brownian motion on M cannot be recurrent.

Thus, we have established that Brownian motion on M is transient if and only if it is not recurrent. This gives the desired result. ☐

The rest of this section is devoted to proving Proposition 5.4.

Let D be an open subset of M and let K be a closed subset. The first exit time τ_D from D and the first hitting time T_K of K are defined by

$$\begin{aligned} \tau_D &= \inf\{t < e \colon X_t \notin D\} \ , \quad \text{and} \\ T_K &= \inf\{t < e \colon X_t \in K\} \ , \end{aligned}$$

where we agree to set $\inf \emptyset = e$. We note that τ_D and T_K are stopping times because both K and $M \setminus D$ are closed subsets of M.

The next lemma states that Brownian motion on M almost surely leaves a relatively compact and non-dense domain D with smooth boundary in finite time.

Lemma 5.6 *Let $D \subset M$ be a relatively compact domain with smooth boundary and such that $M \setminus \overline{D} \neq \emptyset$. It holds true that*

$$\sup_{x \in D} \mathbb{E}_x \tau_D < \infty \ .$$

Proof. Let $x \in D$ be arbitrary. From Proposition 4.17 we recall the identity

$$\mathbb{P}_x(X_t \in C, t < \tau_D) = \int_C p_D(t, x, y) \, \mathrm{d}y \ , \tag{5.2}$$

for $t > 0$ and any Borel subset C of M. Let $\alpha \in \mathbb{R}$ be given by

$$\alpha = \sup_{z \in D} \int_D p_D(1, z, y) \, \mathrm{d}y \; .$$

Using (5.2) we deduce

$$\mathbb{P}_x(\tau_D > 1) = \mathbb{P}_x(X_1 \in D, 1 < \tau_D) = \int_D p_D(1, x, y) \, \mathrm{d}y \leq \alpha \; . \tag{5.3}$$

From Theorem 4.14 (iv) and (v) it follows that $\int_D p_D(t, z, y) \, \mathrm{d}y = 0$ for all $z \in \partial D$ and $t > 0$. Moreoever, $\int_D p_D(1, z, y) \, \mathrm{d}y$ is continuous on $\overline{D}$. Thus, since $M \setminus \overline{D} \neq \emptyset$ by assumption and as $\overline{D}$ is compact, Theorem 4.14 (vii) implies that

$$\alpha = \sup_{z \in \overline{D}} \int_D p_D(1, z, y) \, \mathrm{d}y < 1 \; .$$

Using the Chapman-Kolmogorov equation, cf. Theorem 4.14 (vi), one further obtains that for $n \in \mathbb{N}$ with $n \geq 2$

$$\begin{aligned}
\mathbb{P}_x(\tau_D > n) &= \int_D p_D(n, x, y) \, \mathrm{d}y = \int_D \left(\int_D p_D(n-1, x, z) p_D(1, z, y) \, \mathrm{d}z \right) \mathrm{d}y \\
&= \int_D p_D(n-1, x, z) \left(\int_D p_D(1, z, y) \, \mathrm{d}y \right) \mathrm{d}z \\
&\leq \alpha \int_D p_D(n-1, x, z) \, \mathrm{d}z = \alpha \, \mathbb{P}_x(\tau_D > n-1) \; .
\end{aligned}$$

We were allowed to interchange the order of integration since $p_D(t, x, y)$ is non-negative everywhere so that Fubini's theorem applies. From (5.3) it then follows by induction that for $n \in \mathbb{N}$

$$\mathbb{P}_x(\tau_D > n) \leq \alpha^n \; .$$

Since it certainly holds true that $\mathbb{P}_x(\tau_D > 0) \leq 1$, we deduce

$$\mathbb{E}_x \tau_D \leq \sum_{n=0}^{\infty} \mathbb{P}_x(\tau_D > n) \leq \sum_{n=0}^{\infty} \alpha^n \; .$$

Finally, $\alpha < 1$ implies that

$$\sup_{x \in D} \mathbb{E}_x \tau_D \leq \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} < \infty \; ,$$

as claimed. □

Let $D \subset M$ be a relatively compact domain with smooth boundary ∂D and let f be a continuous function on ∂D. From Hörmander [8, Chapter 20] we recall that there exists a unique function $u \in C^2(D) \cap C\left(\overline{D}\right)$ solving the Dirichlet problem $\Delta_M u(x) = 0$ for all $x \in D$ and $u(x) = f(x)$ for all $x \in \partial D$. The uniqueness of this solution follows from the maximum principle. In fact, we have the following result.

Lemma 5.7 *Let D and f be as above with the additional assumption that $M \setminus \overline{D} \neq \emptyset$. The unique solution u of the corresponding Dirichlet problem is given by*

$$u(x) = \mathbb{E}_x f\left(X_{\tau_D}\right) \; ,$$

where X is a Brownian motion on M starting at $x \in \overline{D}$.

Proof. If $x \in \partial D$ then $\tau_D = 0$ and $\mathbb{E}_x f(X_{\tau_D}) = \mathbb{E}_x f(X_0) = f(x) = u(x)$, as claimed.

Otherwise, we have $x \in D$. Let $t > 0$ be fixed. We observe that $X_r \in D$ for all $r \in [0, t \wedge \tau_D)$ and therefore, $\Delta_M u(X_r) = 0$. By examining the proof of Theorem 4.7, we note it also establishes that X being a Brownian motion on M implies that

$$f(X_s) - f(X_0) - \frac{1}{2} \int_0^s \Delta_M f(X_r)\, \mathrm{d}r$$

is a local martingale for every twice-continuously differentiable function $f \in C^2(M)$. In particular, since $u \in C^2(D)$ the stochastic process $N = (N_s)_{t \wedge \tau_D > s \geq 0}$ given by

$$N_s = u(X_s) - u(X_0) - \frac{1}{2} \int_0^s \Delta_M u(X_r)\, \mathrm{d}r = u(X_s) - u(x)$$

is a local martingale on $\mathbb{R}$. Moreover, u must be bounded as it is a continuous function on a compact set. It follows that N is a true martingale. Furthermore, we claim that $(N_s)_{t \wedge \tau_D \geq s \geq 0}$ with

$$N_{t \wedge \tau_D} = \lim_{s \uparrow t \wedge \tau_D} (u(X_s) - u(x)) = u(X_{t \wedge \tau_D}) - u(x)$$

is also a true martingale. Indeed, by using the boundedness of u and the dominated convergence theorem, we deduce that

$$\mathbb{E}_x[N_{t \wedge \tau_D} | \mathcal{F}_s] = \mathbb{E}_x \left[\lim_{r \uparrow t \wedge \tau_D, r \geq s} N_r \middle| \mathcal{F}_s \right] = \lim_{r \uparrow t \wedge \tau_D, r \geq s} \mathbb{E}_x[N_r | \mathcal{F}_s] = N_s$$

for $0 \leq s < t \wedge \tau_D$. Since $N_0 = 0$, we obtain $\mathbb{E}_x [N_{t \wedge \tau_D}] = 0$ which yields

$$u(x) = \mathbb{E}_x u(X_{t \wedge \tau_D}) \,.$$

From Lemma 5.6 we also know that $\mathbb{P}_x(\tau_D < \infty) = 1$. Therefore, by letting t tend to $+\infty$ and by using the dominated convergence theorem once again, we conclude

$$u(x) = \mathbb{E}_x u\left(X_{\tau_D}\right) = \mathbb{E}_x f\left(X_{\tau_D}\right) \,,$$

as claimed. □

For $K \in \mathscr{K}(M)$, let h_K be the function on M defined by $h_K(x) = \mathbb{P}_x(T_K < e)$. We call h_K the hitting probability of the set K. It certainly holds true that $h_K(x) = 1$ for all $x \in K$. Further properties of the hitting probability are established below.

Lemma 5.8 *For $K \in \mathscr{K}(M)$, the hitting probability h_K is harmonic on $M \setminus K$, i.e. $\Delta_M h_K \equiv 0$ on $M \setminus K$, and continuous on $\overline{M \setminus K}$.*

Proof. To prove harmonicity of h_K on $M \setminus K$, let us consider some arbitrary $x \in M \setminus K$. Since $M \setminus K$ is open, we can find a geodesic ball B centred at x such that $\overline{B} \subset M \setminus K$. Moreover, by choosing the radius of B small enough we can ensure that its boundary is smooth. As before, let τ_B denote the first exit time from B. By construction, we have $\tau_B < T_K$. In particular, on the event $\{T_K < e\}$ we also have $\tau_B < e$. Applying the strong Markov property at the stopping time τ_B, we then deduce

$$\begin{aligned} h_K(x) = \mathbb{P}_x(T_K < e) = \mathbb{E}_x \left[\mathbb{1}_{\{T_K < e\}}\right] &= \mathbb{E}_x \left[\mathbb{E}_x \left[\mathbb{1}_{\{T_K < e\}} \middle| \mathcal{F}_{\tau_B}\right]\right] \\ &= \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_B}} \left[\mathbb{1}_{\{T_K < e\}}\right]\right] \\ &= \mathbb{E}_x \left[\mathbb{P}_{X_{\tau_B}} \left(T_K < e\right)\right] \\ &= \mathbb{E}_x h_K\left(X_{\tau_B}\right) \,. \end{aligned}$$

In Hsu [9, Proof of Proposition 4.4.4], it is stated that this and Lemma 5.7 imply that h_K is the solution of a Dirichlet problem on B. However, I was not able to work out how one could apply Lemma 5.7 as we do not yet know that h_K is continuous on ∂B. If it was, we could argue in a way similar to the one below. On the other hand, if h_K is indeed the solution of a Dirichlet problem on B then this yields the harmonicity of h_K on $M \setminus K$ as $x \in M \setminus K$ was arbitrary. In the following, we shall assume that h_K is harmonic on $M \setminus K$ and present the rest of the proof given in Hsu [9, Chapter 4.4].

If h_K is harmonic on $M \setminus K$ then we also know that h_K is continuous on $M \setminus K$. Therefore, it remains to establish the continuity of h_K at the boundary of $M \setminus K$. Let $(D_n)_{n\in\mathbb{N}}$ be an exhaustion sequence of M with the additional property that $K \subset D_n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, one defines a function u_n on $\overline{D_n \setminus K}$ by

$$u_n(x) = \mathbb{P}_x(T_K < \tau_{D_n}) \ .$$

Since K is particularly closed, we have $\partial K \subset K$, whereas D_n being open implies $\partial D_n \not\subset D_n$. Hence, it holds true that

$$u_n(x) = \begin{cases} 0 & \text{if } x \in \partial D_n \ , \\ 1 & \text{if } x \in \partial K \ . \end{cases} \tag{5.4}$$

We note that $\tau_{D_n} \wedge T_K = \tau_{D_n \setminus K}$. Since $K \in \mathscr{K}(M)$ has non-empty interior, we also have

$$M \setminus \left(\overline{D_n \setminus K}\right) \neq \emptyset \ .$$

Thus, from Lemma 5.6 we obtain $\mathbb{P}_x\left(\tau_{D_n} \wedge T_K < \infty\right) = 1$. Therefore, we can apply the strong Markov property at the finite stopping time $\tau_{D_n} \wedge T_K$ to deduce that

$$u_n(x) = \mathbb{P}_x(T_K < \tau_{D_n}) = \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_{D_n}\wedge T_K}}\left(T_K < \tau_{D_n}\right)\right] = \mathbb{E}_x u_n\left(X_{\tau_{D_n}\wedge T_K}\right) \ . \tag{5.5}$$

On each connected component of $D_n \setminus K$, we now consider the Dirichlet problem with the boundary condition given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \partial D_n \ , \\ 1 & \text{if } x \in \partial K \ . \end{cases}$$

Since $M \setminus \left(\overline{D_n \setminus K}\right) \neq \emptyset$ and as f is continuous, it follows by Lemma 5.7 and (5.4) as well as (5.5) that the unique solution to this Dirichlet problem is

$$\mathbb{E}_x f\left(X_{\tau_{D_n\setminus K}}\right) = \mathbb{E}_x u_n\left(X_{\tau_{D_n}\wedge T_K}\right) = u_n(x) \ .$$

In particular, u_n must be continuous on $\overline{D_n \setminus K}$. By setting $u_n(x) = 0$ whenever $x \in M \setminus \overline{D_n}$ we extend u_n to a continuous function on $\overline{M \setminus K}$. Since $D_n \subset D_{n+1}$ for each $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} D_n = M$, we further conclude that $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence with limit h_K restricted to $\overline{M \setminus K}$. Thus, h_K is lower semicontinuous on $\overline{M \setminus K}$. Therefore, for any $x \in \partial K$, we have

$$\liminf_{y\to x, y\in\overline{M\setminus K}} h_K(y) \geq h_K(x) = 1 \ .$$

Since it also holds true that $h_K(y) \leq 1$ for all $y \in M$, we obtain

$$\lim_{y\to x, y\in\overline{M\setminus K}} h_K(y) = h_K(x) \ ,$$

which establishes the continuity of h_K at the boundary of $\overline{M \setminus K}$. □

Corollary 5.9 *For any $K \in \mathscr{K}(M)$ we have either $h_K \equiv 1$ on M or $K \neq M$ and $0 < h_K(x) < 1$ for all $x \in M \setminus K$.*

Proof. If $K = M$ it is certainly the case that $h_K \equiv 1$ on M. Therefore, in the following, we may assume that $M \setminus K$ is non-empty.

From the definition $h_K(x) = \mathbb{P}_x(T_K < e)$, it follows that $0 \leq h_K(x) \leq 1$ for all $x \in M$. Suppose we have $h_K(x) = 1$ for some $x \in M \setminus K$. Since h_K is harmonic on $M \setminus K$ the maximum principle implies that $h_K(x) = 1$ for all $x \in M \setminus K$. Due to $h_K(x) = 1$ for $x \in K$ it holds true that $h_K \equiv 1$ on M and we are in the first case.

If there does not exist some $x \in M \setminus K$ with $h_K(x) = 1$ then we have $h_K(x) < 1$ for all $x \in M \setminus K$. It remains to exclude the case $h_K(x) = 0$. Indeed, if we could find some $x \in M \setminus K$ with $h_K(x) = 0$ then the minimum principle, i.e. the maximum principle applied to the function $-h_K$, yields $h_K \equiv 0$ on $M \setminus K$. However, since $h_K(x) = 1$ for $x \in K$, this contradicts the continuity of h_K on $\overline{M \setminus K}$. Thus, we must have $h_K(x) > 0$ for all $x \in M \setminus K$ and we are in the second case. □

The next lemma shows that these two cases correspond to $K \in \mathscr{K}(M)$ being recurrent or transient, respectively. In particular, this establishes the first part of Proposition 5.4.

Lemma 5.10 *A set $K \in \mathscr{K}(M)$ is recurrent if $h_K \equiv 1$ on M, whereas it is transient if $K \neq M$ and $0 < h_K(x) < 1$ for all $x \in M \setminus K$.*

Proof. First, suppose that $h_K \equiv 1$ on M and consider the event

$$R = \{K \text{ is recurrent for } X\} .$$

We aim to prove that $\mathbb{P}_x(R) = 1$ for all $x \in M$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of stopping times strictly increasing to the explosion time e. For instance, if M is compact, in which case we have $e = \infty$, one can take $\xi_n = n$. On the other hand, if M is not compact one could consider an appropriate exhaustion sequence $(D_n)_{n\in\mathbb{N}}$ and set $\xi_n = \tau_{D_n}$. If K is recurrent for a path $X(\omega)$ then for every $n \in \mathbb{N}$ there exists some $t_n \geq \xi_n(\omega)$ such that $X_{t_n}(\omega) \in K$. Thus, if we set

$$R_n = \{\exists\, t \geq \xi_n \text{ such that } X_t \in K\}$$

then $R \subset \cap_{n=1}^{\infty} R_n$. On the other hand, if $X(\omega) \in R_n$ for all $n \in \mathbb{N}$ then $X(\omega) \in R$ by Definition 5.1. Hence, we obtain

$$R = \bigcap_{n=1}^{\infty} R_n .$$

Since $R_n \supset R_{n+1}$ for all $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} \mathbb{P}_x(R_n) = \mathbb{P}_x(R)$$

and therefore, it suffices to establish $\mathbb{P}_x(R_n) = 1$ for every $n \in \mathbb{N}$. By using the strong Markov property at the finite stopping time ξ_n we deduce that

$$\mathbb{P}_x(R_n) = \mathbb{E}_x\left[\mathbb{P}_{X_{\xi_n}}(T_K < e)\right] = \mathbb{E}_x h_K\left(X_{\xi_n}\right) .$$

By assumption $h_K \equiv 1$ on M and thus, $\mathbb{P}_x(R_n) = 1$, as required.

Secondly, we deal with the case where $K \neq M$ and $0 < h_K(x) < 1$ for all $x \in M \setminus K$. We aim to prove that K is transient. By (5.1) it suffices to establish $\mathbb{P}_x(R) = 0$ for all $x \in M$, where R is the same event as considered above. Due to $M \setminus K \neq \emptyset$ we can find a relatively compact domain D with smooth and non-empty boundary ∂D such that $M \setminus \overline{D} \neq \emptyset$ and $K \subset D$. By construction, we have $\partial D \subset M \setminus K$ and hence, $h_K(x) < 1$ for all $x \in \partial D$. Since the boundary of a relatively compact domain is compact, it follows that

$$\alpha = \sup_{x\in\partial D} h_K(x) < 1 .$$

We now consider two sequences of stopping times $(\zeta_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ which are given by $\zeta_1 = 0$ and

$$\sigma_n = \inf\{t \geq \zeta_n \colon X_t \in K\} ,$$
$$\zeta_{n+1} = \inf\{t \geq \sigma_n \colon X_t \notin D\}$$

for $n \in \mathbb{N}$. As before, we agree to set $\inf \emptyset = e$. By Lemma 5.6 we have $\tau_D < \infty$ a.s. Moreover, if X explodes in finite time then it leaves every relatively compact and non-dense domain before explosion. It follows that $\tau_D < e$ a.s. Thus, for any $n \in \mathbb{N}$ we obtain

$$R \subset \{\sigma_n < e\}$$

up to null sets and therefore,

$$\mathbb{P}_x(R) \leq \mathbb{P}_x(\sigma_n < e) . \tag{5.6}$$

Let $n \geq 2$. On the event $\{\zeta_n < e\}$, we have $X_{\zeta_n} \in \partial D$ and can use the strong Markov property to deduce that

$$\begin{aligned}
\mathbb{P}_x(\sigma_n < e) &= \mathbb{E}_x\left[\mathbb{P}_{X_{\zeta_n}}(T_K < e)\mathbb{1}_{\{\zeta_n < e\}}\right] \\
&= \mathbb{E}_x\left[h_K(X_{\zeta_n})\mathbb{1}_{\{\zeta_n < e\}}\right] \\
&\leq \alpha\, \mathbb{P}_x(\zeta_n < e) .
\end{aligned}$$

Furthermore, we observe that $\{\zeta_n < e\} \subset \{\sigma_{n-1} < e\}$. Thus,

$$\mathbb{P}_x(\sigma_n < e) \leq \alpha\, \mathbb{P}_x(\sigma_{n-1} < e)$$

and as we certainly have $\mathbb{P}_x(\sigma_1 < e) \leq 1$, it follows by induction that $\mathbb{P}_x(\sigma_n < e) \leq \alpha^{n-1}$. Therefore, by (5.6) it holds true that

$$\mathbb{P}_x(R) \leq \mathbb{P}_x(\sigma_n < e) \leq \alpha^{n-1} ,$$

for all $n \in \mathbb{N}$. This indeed implies $\mathbb{P}_x(R) = 0$ because $\alpha^{n-1} \to 0$ as $n \to \infty$ for $\alpha < 1$. □

It remains to prove the second part of Proposition 5.4.

Lemma 5.11 *If there exists some $L \in \mathscr{K}(M)$ which is recurrent then any set $K \in \mathscr{K}(M)$ is recurrent.*

Proof. We are given the recurrent set $L \in \mathscr{K}(M)$. Let $K \in \mathscr{K}(M)$ be arbitrary and let

$$\alpha = \inf_{x \in L} h_K(x) .$$

By Corollary 5.9, we have $0 < h_K(x) \leq 1$ for all $x \in M$ and the compactness of L yields $\alpha > 0$. As in the first part of the previous proof, we consider a sequence of stopping times $(\xi_n)_{n\in\mathbb{N}}$ strictly increasing to the explosion time e. For $n \in \mathbb{N}$, we again set

$$R_n = \{\exists\, t \geq \xi_n \text{ such that } X_t \in K\}$$

and conclude $\mathbb{P}_x(R_n) = \mathbb{E}_x\left[\mathbb{P}_{X_{\xi_n}}(T_K < e)\right]$. Let $\theta = \inf\{t \geq T_L \colon X_t \in K\}$. We observe that for any $y \in M$

$$\mathbb{P}_y(T_K < e) \geq \mathbb{P}_y(\theta < e) . \tag{5.7}$$

By assumption, L is recurrent and therefore $h_L \equiv 1$ on M. As before, using the strong Markov property at the finite stopping time T_L gives

$$\mathbb{P}_y(\theta < e) = \mathbb{E}_y\left[\mathbb{P}_{X_{T_L}}(T_K < e)\right] = \mathbb{E}_y h_K(X_{T_L}) .$$

Moreover, since L is particularly closed and as T_L is finite we also have $X_{T_L} \in L$. By (5.7) it follows that

$$\mathbb{P}_y(T_K < e) \geq \mathbb{E}_y h_K(X_{T_L}) \geq \alpha \ .$$

As $y \in M$ was arbitrary this yields

$$\mathbb{P}_x(R_n) = \mathbb{E}_x\left[\mathbb{P}_{X_{\xi_n}}(T_K < e)\right] \geq \alpha$$

for every $n \in \mathbb{N}$. Thus, for $R = \{K \text{ is recurrent for } X\}$ and every $x \in M$ we obtain

$$\mathbb{P}_x(R) = \lim_{n\to\infty} \mathbb{P}_x(R_n) \geq \alpha > 0 \ .$$

By (5.1) this implies that K cannot be transient. As any set in $\mathscr{K}(M)$ is either recurrent or transient, K must be recurrent. □

5.2 Equivalent criterions for transience

In this section, we find two conditions which are equivalent to Brownian motion on M being transient. Since Brownian motion on M is either recurrent or transient, cf. Corollary 5.5, those conditions can be turned into criterions for the recurrence of Brownian motion.

One of the two conditions makes use of the function

$$G_M(x, y) = \frac{1}{2}\int_0^\infty p_M(t, x, y)\,\mathrm{d}t$$

for $x, y \in M$ and where $p_M(t, x, y)$ is the minimal heat kernel from Section 4.3. Since $p_M(t, x, y)$ is strictly positive on $(0, \infty) \times M \times M$, cf. Theorem 4.19 (i), the function $G_M(x, y)$ is well-defined for all $x, y \in M$, provided we allow it to take the value infinity. We generally call $G_M(x, y)$ the Green function on M because as stated in Grigor'yan [5, Chapter 4.2], it is the smallest positive fundamental solution of the Laplace equation on M.

The next theorem is part of a theorem in Grigor'yan [5, Chapter 5] which contains a lot more conditions equivalent to the transience of Brownian motion on M. We follow Hsu [9, Chapter 4.4] for the proof of the equivalence of (i) and (ii), whereas we use Grigor'yan [5, Chapter 5] and Davies [1] to prove that (ii) is equivalent to (iii).

Theorem 5.12 *Let M be a Riemannian manifold. The following are equivalent.*

(i) *Brownian motion on M is transient.*

(ii) *For all/some $x \neq y$, the Green function on M is finite, i.e. $G_M(x, y) < \infty$.*

(iii) *For all/some $z \in M$, it holds true that $\int_1^\infty p_M(t, z, z)\,\mathrm{d}t < \infty$.*

We read this theorem in the way that whenever there is a choice between 'all' and 'some' we choose the one which makes the corresponding statement stronger. For instance, (i) ⇒ (ii) states that Brownian motion on M being transient implies that $G_M(x, y)$ is finite for all $x \neq y$. Similarly, (ii) ⇒ (i) says that the existence of some $x \neq y$ with $G_M(x, y) < \infty$ implies the transience of Brownian motion on M.

Proof of (i) ⇔ (ii). We prove the 'if' direction by establishing the contrapositive, i.e. we assume that Brownian motion on M is recurrent and aim to show that $G_M(x, y) = \infty$ for all $x \neq y$. Let $K \in \mathscr{K}(M)$ be small enough so that there exists some relatively compact and non-dense domain $D \subset M$ with smooth boundary and such that $K \subset D$. Moreover, let $x \neq y$ be fixed. Using the non-negativity of $p_M(t, x, y)$ and the Chapman-Kolmogorov equation, cf. Theorem 4.19 (i) and (v), we deduce

$$G_M(x, y) \geq \frac{1}{2}\int_0^\infty p_M(t+1, x, y)\,\mathrm{d}t = \frac{1}{2}\int_0^\infty \left(\int_M p_M(t, x, z)p_M(1, z, y)\,\mathrm{d}z\right)\mathrm{d}t \ . \tag{5.8}$$

For the fixed $y \in M$, we define
$$\alpha = \inf_{z \in K} p_M(1, z, y) \ .$$
Since $p_M(1, z, y)$ is strictly positive as well as continuous on M and as K is compact, it follows that $\alpha > 0$. From (5.8) and Proposition 4.18, we further obtain that

$$\begin{aligned} G_M(x, y) &\geq \frac{\alpha}{2} \int_0^\infty \left(\int_K p_M(t, x, z) \, \mathrm{d}z \right) \mathrm{d}t \\ &= \frac{\alpha}{2} \int_0^\infty \mathbb{P}_x(X_t \in K, t < e) \, \mathrm{d}t = \frac{\alpha}{2} \, \mathbb{E}_x \int_0^\infty \mathbb{1}_{\{X_t \in K, t < e\}} \, \mathrm{d}t \ . \end{aligned}$$

In the last step, we applied Fubini's Theorem. As before, let τ_D be the first exist time from D and let $(\zeta_n)_{n \in \mathbb{N}}$, $(\sigma_n)_{n \in \mathbb{N}}$ be the sequences of stopping times given by $\zeta_1 = 0$ as well as

$$\begin{aligned} \sigma_n &= \inf\{t \geq \zeta_n \colon X_t \in K\} \ , \\ \zeta_{n+1} &= \inf\{t \geq \sigma_n \colon X_t \notin D\} \end{aligned}$$

for $n \in \mathbb{N}$. We claim that the recurrence of K implies that $\sigma_n < e$ and $\zeta_n < e$ for all $n \in \mathbb{N}$. First, we note $\mathbb{P}_x(T_K < e) = h_K(x) = 1$ which yields $\sigma_1 = T_K < e$. Secondly, by Lemma 5.6 and the fact that if X explodes in finite time then it leaves every relatively compact and non-dense domain before explosion, we also have $\tau_D < e$. Thus, if $\sigma_m < e$ for some fixed $m \in \mathbb{N}$, we can apply the strong Markov property at the finite stopping time σ_m to deduce $\zeta_{m+1} < e$. By using the strong Markov property at the finite stopping time ζ_{m+1} and due to the recurrence of K we conclude $\sigma_{m+1} < e$ from $\zeta_{m+1} < e$. The desired result then follows by induction. In particular, $\zeta_n \leq t \leq \zeta_{n+1}$ ensures $t < e$. Moreover, we observe that $\mathbb{1}_{\{X_t \in K\}} = 0$ for all t with $\zeta_n < t < \sigma_n$. Therefore, by also using the strong Markov property at the finite stopping times σ_n we deduce

$$\begin{aligned} \mathbb{E}_x \int_0^\infty \mathbb{1}_{\{X_t \in K, t < e\}} \, \mathrm{d}t &\geq \sum_{n=1}^\infty \mathbb{E}_x \int_{\zeta_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t \in K\}} \, \mathrm{d}t \\ &= \sum_{n=1}^\infty \mathbb{E}_x \int_{\sigma_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t \in K\}} \, \mathrm{d}t \\ &= \sum_{n=1}^\infty \mathbb{E}_x \mathbb{E}_{X_{\sigma_n}} \int_0^{\tau_D} \mathbb{1}_{\{X_t \in K\}} \, \mathrm{d}t \ . \end{aligned}$$

From Proposition 4.17, it follows that for any $u \in K$

$$\mathbb{E}_u \int_0^{\tau_D} \mathbb{1}_{\{X_t \in K\}} \, \mathrm{d}t = \mathbb{E}_u \int_0^\infty \mathbb{1}_{\{X_t \in K, t < \tau_D\}} \, \mathrm{d}t = \int_0^\infty \left(\int_K p_D(t, u, z) \, \mathrm{d}z \right) \mathrm{d}t \ .$$

By the compactness and non-emptiness of K and due to Theorem 4.14 (i), we get

$$\beta = \inf_{u \in K} \int_0^\infty \left(\int_K p_D(t, u, z) \, \mathrm{d}z \right) \mathrm{d}t > 0 \ .$$

Finally, we observe that $X_{\sigma_n} \in K$. Putting everything together and using $\alpha, \beta > 0$, we obtain

$$G_M(x, y) \geq \frac{\alpha}{2} \sum_{n=1}^\infty \mathbb{E}_x \mathbb{E}_{X_{\sigma_n}} \int_0^{\tau_D} \mathbb{1}_{\{X_t \in K\}} \, \mathrm{d}t \geq \frac{\alpha}{2} \sum_{n=1}^\infty \beta = \infty \ .$$

This yields $G_M(x, y) = \infty$, as required.

For the 'only if' direction, assume that Brownian motion on M is transient. We aim to prove that $G_M(x, y) < \infty$ for all $x \neq y$. Let $x \in M$ be arbitrary and let $K \in \mathscr{K}(M)$ be a small set for which one can find a relatively compact and non-dense domain D with smooth and non-empty boundary such that $K \subset D$. Moreover, let $(\zeta_n)_{n\in N}$ and $(\sigma_n)_{n\in N}$ be the same sequences of stopping times as defined above. Making similar deductions as in the first part of the proof, we obtain

$$2\int_K G_M(x,z)\,\mathrm{d}z = \mathbb{E}_x \int_0^\infty \mathbb{1}_{\{X_t\in K, t<e\}}\,\mathrm{d}t = \mathbb{E}_x \int_0^e \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t = \sum_{n=1}^\infty \mathbb{E}_x \int_{\sigma_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t\ .$$

By the strong Markov property, we also get

$$\mathbb{E}_x \int_{\sigma_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t = \mathbb{E}_x\left[\left(\mathbb{E}_{X_{\sigma_n}}\int_0^{\tau_D} \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t\right)\mathbb{1}_{\{\sigma_n<e\}}\right] \le \mathbb{E}_x\left[\left(\mathbb{E}_{X_{\sigma_n}}\tau_D\right)\mathbb{1}_{\{\sigma_n<e\}}\right]\ . \tag{5.9}$$

Set

$$\gamma = \sup_{z\in D} \mathbb{E}_z \tau_D\ ,$$

which has to satisfy $\gamma < \infty$ by Lemma 5.6. Since $X_{\sigma_n} \in K \subset D$ we can use (5.9) to further deduce that

$$\mathbb{E}_x \int_{\sigma_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t \le \gamma\,\mathbb{E}_x\left[\mathbb{1}_{\{\sigma_n<e\}}\right] = \gamma\,\mathbb{P}_x(\sigma_n<e)\ .$$

On the other hand, as in the second half of the proof of Lemma 5.10 we also have $\mathbb{P}_x(\sigma_n < e) \le \delta^{n-1}$, where

$$\delta = \sup_{z\in\partial D} h_K(z)\ .$$

Moreover, as before it holds true that $\delta < 1$ because K is transient and ∂D is compact. In total, it follows that

$$\begin{aligned} 2\int_K G_M(x,z)\,\mathrm{d}z &= \sum_{n=1}^\infty \mathbb{E}_x \int_{\sigma_n}^{\zeta_{n+1}} \mathbb{1}_{\{X_t\in K\}}\,\mathrm{d}t \le \gamma\sum_{n=1}^\infty \mathbb{P}_x(\sigma_n<e) \\ &\le \gamma\sum_{n=1}^\infty \delta^{n-1} = \frac{\gamma}{1-\delta} < \infty\ . \end{aligned}$$

In particular, there must exist some $y \in K$ with $y \neq x$ and such that $G_M(x,y) < \infty$. Due to the implication (ii) $\Rightarrow$ (iii), which we prove below, this implies $\int_1^\infty p_M(t,z,z)\,\mathrm{d}t < \infty$ for all $z \in M$ and hence, by (iii) $\Rightarrow$ (ii) it follows that $G_M(x,y) < \infty$ for all $y \neq x$. □

Proof of (ii) $\Leftrightarrow$ (iii). We need the following fact, which is a special case of [1, Theorem 10]. If $T > 0$ and $x_1, x_2, x_3, x_4 \in M$ are fixed then there exists a positive constant $c = c(T, x_1, x_2, x_3, x_4)$ such that

$$p_M(t,x_1,x_2) \le c\,p_M(t,x_3,x_4)$$

for all $t \ge T$. A proof which makes use of the Chapman-Kolmogorov equation and the local parabolic Harnack inequality is given in Davies [1].

Let us assume that there exists some $x \neq y$ in M with $G_M(x,y) < \infty$. Due to

$$G_M(x,y) = \frac{1}{2}\int_0^\infty p_M(t,x,y)\,\mathrm{d}t$$

it follows that

$$\int_1^\infty p_M(t,x,y)\,\mathrm{d}t < \infty\ .$$

By the fact given, for any $z \in M$ there exists some positive constant c such that

$$p_M(t,z,z) \le c\,p_M(t,x,y)$$

for all $t \geq 1$. In particular, we obtain

$$\int_1^\infty p_M(t,z,z)\,\mathrm{d}t \leq c \int_1^\infty p_M(t,x,y)\,\mathrm{d}t < \infty\ .$$

As $z \in M$ was arbitrary, this gives (iii).

Conversely, assume that $\int_1^\infty p_M(t,z,z)\,\mathrm{d}t < \infty$ for some fixed $z \in M$. Let $x \neq y$ be arbitrary. By again using the above fact, we deduce

$$\int_1^\infty p_M(t,x,y)\,\mathrm{d}t < \infty\ .$$

Furthermore, due to $x \neq y$ and Theorem 4.19 (iii) we have

$$p_M(t,x,y) \to 0 \quad \text{as} \quad t \to 0\ .$$

Thus, it follows that $\int_0^1 p_M(t,x,y)\,\mathrm{d}t < \infty$ which yields

$$G_M(x,y) = \frac{1}{2}\int_0^\infty p_M(t,x,y)\,\mathrm{d}t < \infty\ ,$$

as claimed. □

5.3 Considering concrete examples

Finally, we present some examples of Riemannian manifolds for which one can decide whether Brownian motion on them is recurrent or transient.

First, let us consider the manifold $M = \mathbb{R}^d$ equipped with a metric of the form

$$h = \mathrm{d}r^2 + \sigma(r)^2 h_{S^{d-1}}\ ,$$

where $\sigma(r)$ is a positive function on $(0,\infty)$ and where $h_{S^{d-1}}$ denotes the standard Euclidean metric on the sphere S^{d-1}. For instance, the Euclidean metric on $\mathbb{R}^d$ has $\sigma(r) = r$, whereas the disc model of the hyperbolic plane in geodesic polar coordinates corresponds to the case $d = 2$ and $\sigma(r) = \sinh(r)$.

By using the Riemannian metric h we can define the distance $d(x,y)$ between two points $x, y \in \mathbb{R}^d$. Let $S_h(r)$ denote the area of a geodesic sphere of radius r centred at the origin in the Riemannian manifold $\left(\mathbb{R}^d, h\right)$. In Grigor'yan [5, Example 4.1] it is shown that the Green function $G_{(\mathbb{R}^d,h)}$ on $\left(\mathbb{R}^d, h\right)$ satisfies

$$G_{(\mathbb{R}^d,h)}(0,x) = \int_{d(0,x)}^\infty \frac{\mathrm{d}r}{S_h(r)}\ .$$

We observe that a geodesic sphere of radius r centred at the origin in the Riemannian manifold $\left(\mathbb{R}^d, h\right)$ corresponds to a $(d-1)$-dimensional Euclidean sphere of radius $\sigma(r)$. Thus, if c_{d-1} is the area of a $(d-1)$-dimensional Euclidean unit sphere then

$$S_h(r) = \sigma(r)^{d-1} c_{d-1}\ .$$

It follows that

$$G_{(\mathbb{R}^d,h)}(0,x) = \int_{d(0,x)}^\infty \frac{\mathrm{d}r}{\sigma(r)^{d-1} c_{d-1}}\ . \tag{5.10}$$

We recall that for $a > 0$

$$\int_a^\infty \frac{\mathrm{d}r}{r^{d-1}} = \infty \quad \text{if} \quad d \leq 2 \qquad \text{and} \qquad \int_a^\infty \frac{\mathrm{d}r}{r^{d-1}} < \infty \quad \text{if} \quad d > 2\ .$$

Since $d(0, x) > 0$ if $x \neq 0$, we can use (5.10) and Theorem 5.12 to verify that Brownian motion on the Euclidean space $\mathbb{R}^d$ is recurrent if $d = 1, 2$ and transient if $d \geq 3$. Furthermore, one can show that for $a > 0$

$$\int_a^\infty \frac{\mathrm{d}r}{\sinh(r)} = -\log\left(\tanh\left(\frac{a}{2}\right)\right) < \infty\ .$$

Hence, again by Theorem 5.12, Brownian motion on the hyperbolic plane is transient. As Brownian motion on the Euclidean plane is recurrent and since both the hyperbolic plane and the Euclidean plane are two-dimensional manifolds, we observe that the recurrence and transience behaviour of Brownian motion on a Riemannian manifold really does depend on its shape and not only on its dimension.

By using the next lemma, it is also easy to give examples of Riemannian manifolds of arbitrary large dimension on which Brownian motion is recurrent.

Proposition 5.13 *Brownian motion on a compact Riemannian manifold M is recurrent.*

Proof. Suppose Brownian motion on M is not recurrent. As in the proof of Corollary 5.5 this implies that any compact subset of M is transient. However, this is a contradiction as M itself is both recurrent and compact. $\square$

Hence, the unit sphere S^d is an example of a d-dimensional manifold on which Brownian motion is recurrent.

References

[1] E. B. Davies: *Non-Gaussian Aspects of Heat Kernel Behaviour.* J. Lond. Math. Soc., II. Ser. 55, No. 1, 105-125 (1997)

[2] K. D. Elworthy: *Stochastic Differential Equations on Manifolds.* London Mathematical Society Lecture Note Series. Vol. 70. CUP (1982)

[3] M. Emery: *Stochastic Calculus in Manifolds.* Universitext. Springer (1989)

[4] R. Feres: *Geometry and Probability.* Lecture Notes Washington University in St. Louis. Math 545 (2001)
URL http://www.math.wustl.edu/~feres/545.pdf (accessed 17/04/2013)

[5] A. Grigor'yan: *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds.* Bull. Am. Math. Soc., New Ser. 36, No. 2, 135-249 (1999)

[6] A. Grigor'yan: *Heat Kernel and Analysis on Manifolds.* AMS/IP Studies in Advanced Mathematics. Vol. 47. AMS (2009)

[7] P. Hänggi: *100 years of Brownian motion: historical items and surveys.*
URL http://www.physik.uni-augsburg.de/theo1/hanggi/History/BM-History.html
(accessed 17/04/2013)

[8] L. Hörmander: *The Analysis of Linear Partial Differential Operators III.* Pseudo-Differential Operators. Grundlehren der mathematischen Wissenschaften. Vol. 274. Springer (1985)

[9] E. H. Hsu: *Stochastic Analysis on Manifolds.* Graduate Studies in Mathematics. Vol. 38. AMS (2002)

[10] E. Jørgensen: *The Central Limit Problem for Geodesic Random Walks.* Z. Wahrscheinlichkeitstheor. Verw. Geb. 32, 1-64 (1975)

[11] I. Karatzas, S. E. Shreve: *Brownian Motion and Stochastic Calculus.* Graduate Texts in Mathematics. Vol. 113. Springer (1991)

[12] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry.* Vol. 1. Interscience Publishers (1963)

[13] J. R. Norris: *A complete differential formalism for stochastic calculus in manifolds.* Séminaire de probabilités XXVI, Lect. Notes Math. 1526, 189-209 (1992)

[14] L. Rogers: *Randomness and Brownian Motion.*
URL http://nrich.maths.org/6127 (accessed 17/04/2013)

[15] L. C. G. Rogers, D. Williams: *Diffusions, Markov Processes and Martingales.* Vol. 1: Foundations. Cambridge Mathematical Library. CUP (2011)

[16] L. C. G. Rogers, D. Williams: *Diffusions, Markov Processes and Martingales.* Vol. 2: Itô Calculus. Cambridge Mathematical Library. CUP (2008)

[17] D. W. Stroock, S. R. S. Varadhan: *Multidimensional Diffusion Processes.* Grundlehren der mathematischen Wissenschaften. Vol. 233. Springer (1979)