Gaussian beams and the propagation of singularities

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1 Introduction

This report deals with linear hyperbolic partial differential equations. The prototype of such an equation is the classical wave equation

$$\Box u := \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right) u = 0 \tag{1.1}$$

for a function u on $\mathbb{R}^{n+1}$. Our notion of hyperbolicity is made precise in the following

Definition 1.1 *Let P be a linear partial differential operator of order m acting on real-valued functions $(t,x) \mapsto u(t,x)$ with $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Setting $f_\alpha := \exp(\mathrm{i}\alpha(t\tau + x \cdot \xi))$, the principal symbol of P is defined by the polynomial*

$$p_m(t,x,\tau,\xi) := \lim_{\alpha \to \infty} \alpha^{-m} [\bar{f}_\alpha P f_\alpha](t,x), \tag{1.2}$$

of degree m. We call P strictly hyperbolic if $s \mapsto p_m(t,x,s,\xi)$ has m distinct real roots for all $\xi \neq 0$.

Remark 1.2 *The coefficient of $\partial^m/\partial t^m$ in all strictly hyperbolic operators of degree m is non-zero. We therefore assume this coefficient to be one.*

It is a feature of the wave equation to allow for travelling wave packets, i. e. for solutions which are localised in space and propagate in time on certain curves and this property is shared by the class of all linear hyperbolic PDEs $Pu = 0$ on $\mathbb{R}^{n+1}$. One can construct such localised solutions by means of Gaussian beams, which are approximate solutions of the form

$$(t,x) \mapsto \mathrm{e}^{\mathrm{i}k\psi(t,x)} \left(a_0(t,x) + \frac{a_1(t,x)}{k} + \ldots + \frac{a_N(t,x)}{k^n} \right), \tag{1.3}$$

where ψ and a_i, for $1 \leq i \leq N \in \mathbb{N}$, are real-valued and $k > 1$. One shows that these Gaussian beams must be concentrated around ray paths.

Definition 1.3 *Let P be a strictly hyperbolic operator and p_m its principal symbol. Let $(\hat{t}, \hat{x}, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ and choose $\hat{\tau} \in \mathbb{R}$ such that*

$$p_m(\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi}) = 0. \tag{1.4}$$

Then $\gamma : s \mapsto (t(s), x(s), \tau(s), \xi(s))$ is a null bicharacteristic curve through $(\hat{t}, \hat{x}, \hat{\xi})$ if the Hamiltonian system

$$\dot{x} = \frac{\partial p_m}{\partial \xi}, \quad \dot{t} = \frac{\partial p_m}{\partial \tau}, \quad \dot{\xi} = -\frac{\partial p_m}{\partial x}, \quad \dot{\tau} = -\frac{\partial p_m}{\partial t} \tag{1.5}$$

is satisfied with $(t(0), x(0), \tau(0), \xi(0)) = (\hat{t}, \hat{x}, \hat{\tau}, \hat{\xi})$. We call the projection of a bicharacteristic on the (t,x)-space a ray path.

Remark 1.4 *Recall that for $\Omega \subseteq \mathbb{R}^{2(n+1)}$ and*

$$F : \Omega \longrightarrow \mathbb{R}, (t,x,\tau,\xi) \mapsto F(t,x,\tau,\xi) \tag{1.6}$$

the Hamiltonian vector field is defined by

$$H_F := \left(\frac{\partial F}{\partial \tau}\frac{\partial}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial}{\partial \tau} \right) + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial \xi_i} \right). \tag{1.7}$$

Thus each null bicharacteristic curve γ satisfies $\dot{\gamma} = H_{p_m}$.

Another characteristic of hyperbolic equations is that they allow for solutions with singularities, which is in sharp contrast to elliptic partial differential equations where a solution u of $Du = f$ for elliptic D and smooth f is always smooth. This gives rise to the interesting question how singularities in solutions to a hyperbolic PDE propagate. The notion of propagation of singularities will be made more precise in due course.

This report is structured as follows. First we will sketch the method of approximating solutions to the linear wave equation using geometrical optics, which is closely related to the Gaussian beam ansatz, and we will explain the limitations of the geometrical method. The subsequent part will be devoted to the construction of Gaussian beams on $\mathbb{R}^{n+1}$ for arbitrary strictly hyperbolic operators P and for the special case of $\Box$. Thereafter we shall explore how to adapt this construction for the background $\mathbb{R} \times \Omega$ with a bounded domain $\Omega \subseteq \mathbb{R}^n$. The last sections will deal with the aforementioned propagation of singularities and will apply the Gaussian beam approximation to characterise the propagation of singularities.

The main reference for the theory presented in this report is Ralston's article [3]. The section about geometrical optics follows the lines of Taylor [5].

2 Geometrical optics

The geometric optics approximation is an important tool in studying the wave equation. The threefold purpose of this section is to introduce this method, to show its limitations and to prepare the way for the Gaussian beam approximation. Therefore we do not intend to give an account of the geometrical optics ansatz in its most general form, but rather presenting an approach parallel to our construction of Gaussian beams.
Consider again the wave equation $\Box u = 0$ on $\mathbb{R}^{n+1}$. We aim to find approximate solutions of the form

$$(t,x) \mapsto \mathrm{e}^{\mathrm{i}k\psi(t,x)} \sum_{j\geq 0} \frac{a_j(t,x)}{(\mathrm{i}k)^j} \tag{2.1}$$

with $a, \psi \in C^\infty((-T,T)\times\mathbb{R}^n)$ for some $T > 0$. More precisely, we want to achieve that, for $v_N := \mathrm{e}^{\mathrm{i}k\psi(t,x)} \sum_{j=0}^N \frac{a_j(t,x)}{(\mathrm{i}k)^j}$,

$$\Box v_N = O(k^{-\nu}) \tag{2.2}$$

in $C^{N+1-\nu}$ for $0 \leq \nu \leq N$. We will see that we can arrange this if, for all N, we require

$$v_N(0,x) = a(x)\,\mathrm{e}^{\mathrm{i}k\varphi(x)} \tag{2.3}$$

where $a \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$ with $\nabla\varphi \neq 0$ on a neighbourhood of $\operatorname{supp} a$.

2.1 Construction of the approximation

One easily calculates that

$$\Box v_N = \mathrm{e}^{\mathrm{i}k\psi}\left(\Box a_0 + \ldots + \frac{\Box a_N}{(\mathrm{i}k)^N}\right) \tag{2.4}$$

$$- 2\mathrm{i}k\,\mathrm{e}^{\mathrm{i}k\psi}\left(\partial_t\psi\left(\partial_t a_0 + \ldots + \frac{\partial_t a_N}{(\mathrm{i}k)^N}\right) - \sum_{i=1}^n \partial_i\psi\left(\partial_i a_0 + \ldots + \frac{\partial_i a_N}{(\mathrm{i}k)^N}\right)\right) \tag{2.5}$$

$$+ \mathrm{i}k\Box\psi v_N + k^2(|\partial_t\psi|^2 - |\nabla_x\psi|^2)v_N \tag{2.6}$$

The coefficient of k^2 is

$$\mathrm{e}^{\mathrm{i}k\psi}\, a_0(|\partial_t\psi|^2 - |\nabla_x\psi|^2), \tag{2.7}$$

the coefficient of k^1 can be written as

$$\mathrm{i}\,\mathrm{e}^{\mathrm{i}k\psi}a_0\Box\psi - \mathrm{i}\,\mathrm{e}^{\mathrm{i}k\psi}(|\partial_t\psi|^2 - |\nabla_x\psi|^2)a_1 \tag{2.8}$$

$$- \mathrm{i}\,\mathrm{e}^{\mathrm{i}k\psi}\left(2\partial_t\psi\frac{\partial a_0}{\partial t} - 2\nabla_x\psi\cdot\nabla_x a_0\right) \tag{2.9}$$

and for k^{1-j} $(j \geq 1)$, we have

$$\mathrm{i}^{1-j}\,\mathrm{e}^{\mathrm{i}k\psi}\, a_j\Box\psi - \mathrm{i}^{1-j}\,\mathrm{e}^{\mathrm{i}k\psi}\, a_{j+1}(|\partial_t\psi|^2 - |\nabla_x\psi|^2) \tag{2.10}$$

$$+ 2\mathrm{i}^{1-j}\,\mathrm{e}^{\mathrm{i}k\psi}\left(\partial_t\psi\partial_t a_1 - \sum_{i=1}^n \partial_i\psi\partial_i a_j\right) + \mathrm{i}^{1-j}\,\mathrm{e}^{\mathrm{i}k\psi}\,\Box a_0. \tag{2.11}$$

We will set these coefficients successively equal to zero. The coefficient of k^2 vanishes if φ satisfies the eikonal equation

$$-\left|\frac{\partial\psi}{\partial t}\right|^2 + |\nabla_x\psi|^2 = 0 \tag{2.12}$$

with initial datum $\psi(0,x) = \varphi(x)$. Below we will prove that there is a neighbourhood U of $K := \operatorname{supp} a$ and a $T > 0$ such that the eikonal equation exhibits a unique solution for each choice of $\sqrt{\cdot}$, i. e. a $\varphi \in C^\infty((-T,T) \times U)$ with

$$\psi(0,x) = \varphi(x), \quad \frac{\partial \psi}{\partial t}(0,x) = -|\nabla_x \psi(x)|. \tag{2.13}$$

Remark 2.1 *Equation (2.12) can be written as*

$$p_2\left(t, x, \frac{\partial \psi}{\partial t}, \nabla_x \psi\right) = 0 \tag{2.14}$$

with the principal symbol p_2 of the wave equation. In the geometrical optics ansatz we require the principal symbol to vanish, but this may lead to a solution which is only local time, i. e. $T < \infty$, meaning that we cannot construct a global approximate solution to the wave equation. In the Gaussian beam approximation, we require equation (2.14) to hold only up to a certain order, which will be crucial to guarantee global approximations.

Proceeding to next order, we see that the coefficient of λ^1 vanishes if the transport equation

$$2\frac{\partial \psi}{\partial t}\frac{\partial a_0}{\partial t} = 2\nabla_x \psi \cdot \nabla_x a_0 + a_0 \Box \psi \tag{2.15}$$

is satisfied. Noting that $\partial_t \varphi \neq 0$ on U for $|t|$ sufficiently small, one shows as in [5] that this PDE has unique solutions once initial conditions are specified. Equation (2.3) implies that

$$a_0(0,x) = a(x). \tag{2.16}$$

Thus we get $a_0 \in C^\infty((-T,T) \times U)$, compactly supported in U on every time slice, for T small enough. The terms in equation (2.10) vanish provided that

$$2\frac{\partial \psi}{\partial t}\frac{\partial a_j}{\partial t} = 2\nabla_x \psi \cdot \nabla_x a_j - a_j \Box \psi + \Box a_{j-1}. \tag{2.17}$$

In the light of (2.3), we require

$$a_j(0,x) = 0. \tag{2.18}$$

Hence the transport equation (2.17) has a unique solution $a_j \in C^\infty((-T,T) \times U)$, compactly supported in U on each time slice, for T small enough.
The above construction yields that

$$\Box v_N = (\mathrm{i}k)^{-N} \Box a_N \, \mathrm{e}^{\mathrm{i}k\psi}, \tag{2.19}$$

and we conclude that

$$\Box v_N = O(k^{-\nu}) \tag{2.20}$$

in $C^{N+1-\nu}((-T,T) \times U)$ for $0 \leq \nu \leq N$.
Therefore the geometrical optics ansatz gives a suitable approximation to solutions of the wave equation. Furthermore, this approximation induces possible initial conditions for an initial value problem in $\mathbb{R}^{n+1}$.

2.2 The eikonal equation

A crucial step in the construction above was the appeal to solutions of the eikonal equation. Considering eikonal equations of the form

$$p_m\left(t, x, \frac{\partial \psi}{\partial t}, \nabla_x \psi\right) = 0 \tag{2.21}$$

to high order is a key idea in the Gaussian beam approximation. To introduce the abstract methods which are related to study this problem, we will give an account of how existence and uniqueness of solutions to the eikonal equation can be shown. We will conclude this section by explaining why solutions to this equation need not be global, which results in a break down of the geometrical optics approximation.
The eikonal equation is of the more general form

$$F(x, \mathrm{d}u) = 0 \tag{2.22}$$

for F smooth on $\mathbb{R}^{2(n+1)}$. Let $S = \{t = 0\}$ and v smooth on S. Require $u|_S = v$. Let $x_0 \in S$ and $\xi_0 := \nabla_x v$. Let $\tau_0 \in \mathbb{R}$ be such that

$$F(x_0, (\tau_0, \xi_0)) = 0. \tag{2.23}$$

Note that if F is the principal symbol of a strictly hyperbolic operator, then such a choice is always possible provided that $\xi_0 \neq 0$. We assume that S satisfies the noncharacteristic hypothesis

$$\frac{\partial F}{\partial \tau}(x_0, (\tau_0, \xi_0)) \neq 0. \tag{2.24}$$

If F is the principal of $\Box$, then this is satisfied as long as $\xi_0 \neq 0$.
We look for a solution by appealing to the theory of Hamiltonian systems. Recall that a symplectic form σ on $\mathbb{R}^{2(n+1)} = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\}$ is given by

$$\sigma = \sum_{j=1}^{n} \mathrm{d}\xi_j \wedge \mathrm{d}x_j. \tag{2.25}$$

Now let Λ be the graph of a function $\xi = \Xi(x)$ in $\mathbb{R}^{2(n+1)}$. Then the following holds.

Proposition 2.2 *The surface Λ is locally the graph of $\mathrm{d}u$ for a smooth function u if and only if*

$$\frac{\partial \Xi_j}{\partial x_k} = \frac{\partial \Xi_k}{\partial x_j} \tag{2.26}$$

for all j, k.

Proof. Condition (2.26) is equivalent to saying that $\sum_{i=1}^{n} \Xi_i(x)\mathrm{d}x_i$ is closed. By the Poincaré lemma, there is a smooth u such that $\mathrm{d}u = \sum_{i=1}^{n} \Xi_i(x)\mathrm{d}x_i$ locally, but this is equivalent to Λ being locally the graph of $\mathrm{d}u$. □

Proposition 2.3 *The surface Λ is locally the graph of $\mathrm{d}u$ if and only if $\sigma(X, Y) = 0$ for all vectors X, Y tangent to Λ.*

Proof. It suffices to check the condition on σ on a generating system of tangent vectors. Define

$$X_j = \frac{\partial}{\partial x_j} + \sum_{l=1}^{n} \frac{\partial \Xi_l}{\partial x_j} \frac{\partial}{\partial \xi_l} \tag{2.27}$$

for $j = 1, \ldots, n$. Then these generate the tangent space and one easily checks that

$$\sigma(X_k, X_j) = \frac{\partial \Xi_j}{\partial x_k} - \frac{\partial \Xi_k}{\partial x_j}. \tag{2.28}$$

The result follows from the previous proposition. □
We define a set $\Sigma \subseteq \mathbb{R}^{2(n+1)}$ over S by

$$\Sigma = \{(x, \xi) \,:\, t = 0,\, \xi_j = \partial_j v,\, F(x, (\tau, \xi)) = 0\} \tag{2.29}$$

using $x = (t, x')$. The noncharacteristic condition implies by the implicit function theorem that there is a local smooth function $\tau(x')$ such that $F(x, (\tau, \xi)) = 0$. Thus Σ is an n-dimensional surface.
Define Λ to be the union of the integral curves of the Hamiltonian vector field H_F through Σ. By the noncharacterisitc condition, H_F has a nonvanishing $\partial/\partial t$ component so that Λ has dimension $n + 1$ and is graph of a function $\xi = \Xi(x)$ in a neighbourhood of x_0.

Theorem 2.4 *The surface Λ is locally the graph of* $\mathrm{d}u$ *for a solution u to*

$$F(x, \mathrm{d}u) = 0, \quad u|_S = 0. \tag{2.30}$$

Proof. Let X, Y be vector fields tangent to Λ at $(x, \xi) \in \Lambda$. By the previous propositions, it suffices to show $\sigma(X, Y) = 0$. First, suppose that $x \in S$, i. e. $(x, \xi) \in \Sigma$. Decompose $X = X_1 + X_2$, $Y = Y_1 + Y_2$ such that X_1, Y_1 are tangent to Σ and X_2, Y_2 are multiples of H_F. The surface Σ is the graph of a gradient if considered as its restriction to $\mathbb{R}^{2n}$ (forgetting t and τ). By Proposition 2.3 we have $\sigma(X_1, Y_1) = 0$. Recalling that $\sigma(H_F, \cdot) = -\mathrm{d}F$ and that $F = 0$ along integral curves of H_F, it follows that

$$\sigma(X, Y) = \sigma(X_1, Y_1) + \sigma(X_2, Y_1,) + \sigma(X_1, Y_2) + \sigma(X_2, Y_2) = 0. \tag{2.31}$$

Denote the flow generated by H_F by $\mathcal{F}^s$. Now let $(x, \xi) \in \Sigma$ and X, Y tangent at $\mathcal{F}^s(x, \xi) \in \Lambda$. We have

$$\sigma(X, Y) = (\mathcal{F}^s)^* \sigma((\mathcal{F}^s)^* X, (\mathcal{F}^s)^* Y), \tag{2.32}$$

where $(\mathcal{F}^2)^*$ denotes the pullback. Note that $(\mathcal{F}^t)^* X, (\mathcal{F}^t)^* Y$ are tangent at Σ. Using that the flow generate by H_F leaves the symplectic form invariant so that

$$\sigma(X, Y) = \sigma((\mathcal{F}^t)^* X, (\mathcal{F}^t)^* Y) = 0, \tag{2.33}$$

which proves the theorem. □

Remark 2.5 *This type of construction using the framework of symplecticity is exactly the one which will be used in the construction of a ψ such that*

$$p_m\left(x, \partial\psi/\partial x\right) = 0. \tag{2.34}$$

The above existence proof of solutions to (2.30) immediately shows why we cannot expect global solutions in general. If integral curves of H_F cross, i. e. if $\mathcal{F}^s$ is not injective anymore, then solutions break down.

Example 2.6 *Let F be the principal symbol of the wave equation in $\mathbb{R}^{2+1}$, i. e. $F(t, x, \tau, \xi) = -\tau^2 + |\xi|^2$. We choose the initial datum $v(x, y) = \sin x + \cos y$ on $S = \{t = 0\}$. Then $\nabla_x v \neq 0$ for $x \in (-\pi/2, \pi/2) \times (-\pi, \pi)$. Defining Σ as above, we act with the Hamiltonian vector field*

$$H_F = 2\sqrt{\cos^2 x + \sin^2 y}\,\frac{\partial}{\partial t} + 2\cos x\frac{\partial}{\partial x} - 2\sin y\frac{\partial}{\partial y}. \tag{2.35}$$

One can easily find two integral curves which cross in finite time.

3 The construction of Gaussian beams

For the pure construction of Gaussian beams, we do not need a distinguished time coordinate as we do not have to think about going forward or backwards in time. Therefore, in order to ease notation, we use $x_0 = t$ until specified otherwise. In particular, we write x to denote $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$.

Let $\mathrm{P}(x, \mathrm{D})$ be a strictly hyperbolic linear partial differential operator of order m with real principal symbol $p_m(x, \xi)$ and let Γ be a smooth curve in $\mathbb{R}^{n+1}$, which is given by $x(s)$ for $s \in \mathbb{R}$. We are interested in finding asymptotic solutions $u(x, k)$ to the partial differential equation $\mathrm{P}(x, \mathrm{D})u = 0$ which become concentrated on Γ as $k \to \infty$. According to the Gaussian beam ansatz, we consider functions u of the form

$$u(x, k) = \mathrm{e}^{\mathrm{i}k\psi(x)} \left(a_0(x) + \frac{a_1(x)}{k} + \cdots + \frac{a_N(x)}{k^N} \right) . \tag{3.1}$$

As we want u to be an asymptotic solution to $\mathrm{P}(x, \mathrm{D})u = 0$ we aim to find $a_0(x), a_1(x), \dots, a_N(x)$ and $\psi(x)$ such that

$$\mathrm{P}(x, \mathrm{D})u = O\left(k^{-M}\right) \tag{3.2}$$

for some large M. On the other hand, we also want u to become concentrated on Γ as $k \to \infty$ and for that reason, we would like to choose $\psi(x)$ such that for all $s \in \mathbb{R}$

(a) $\psi(x(s))$ is real-valued and

(b) $\operatorname{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))$ is positive definite on vectors orthogonal to $\dot{x}(s)$.

We note that if both (a) and (b) are satisfied then u rapidly decreases off Γ because $\left|\mathrm{e}^{\mathrm{i}k\psi(x)}\right|$ looks like a Gaussian distribution with variance proportional to k^{-1} on planes perpendicular to Γ. Thus, after multiplying u by a k-independent function which vanishes outside a small enough neighbourhood of Γ but which is also equal to one on an even smaller neighbourhood of Γ, we obtain an asymptotic solution to $\mathrm{P}(x, \mathrm{D})u = 0$ which indeed becomes concentrated on Γ as $k \to \infty$.

Moreover, the rapid decrease off Γ which is implied by choosing $\psi(x)$ subject to (a) and (b) is also used in establishing (3.2). As we see later, the estimate (3.2) follows from the vanishing of $\mathrm{P}(x, \mathrm{D})u$ on Γ to sufficiently high order.

Applying $\mathrm{P}(x, \mathrm{D})$ to the general form (3.1) gives

$$\mathrm{P}(x, D)u = k^m p_m\left(x, \frac{\partial \psi(x)}{\partial x}\right) \mathrm{e}^{\mathrm{i}k\psi(x)}\, a_0(x) + O(k^{m-1}) .$$

Thus, in order to achieve (3.2) we might want to try to find $\psi(x)$ such that $p_m\left(x, \frac{\partial \psi}{\partial x}\right) \equiv 0$. However, in the geometric optics section, we saw that this could be too much to ask for as it might result in solutions breaking down in finite time. It turns out that it suffices to have $p_m\left(x, \frac{\partial \psi}{\partial x}\right)$ vanish to high order on Γ.

Let us now see what conditions we obtain by requiring that $f(x) = p_m\left(x, \frac{\partial \psi(x)}{\partial x}\right)$ vanishes to high order on Γ. Vanishing of order zero gives

$$p_m\left(x(s), \xi(s)\right) = 0 \tag{3.3}$$

where $\xi(s) = \frac{\partial \psi}{\partial x}(x(s))$, whereas vanishing of order one yields (using summation convention)

$$0 = \frac{\partial f}{\partial x_i} = \frac{\partial p_m}{\partial x_i} + \frac{\partial p_m}{\partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} \tag{3.4}$$

along Γ for $i = 0, 1, \dots, n$. Before we have a look at the higher orders of f, we look at condition (3.4) in more detail.

Since the principal symbol p_m is real, taking the imaginary part of (3.4) gives

$$0 = \frac{\partial p_m}{\partial \xi_k}(x(s), \xi(s)) \left(\operatorname{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_k}(x(s)) \right) .$$

Thus, for all $s \in \mathbb{R}$ we need $\frac{\partial p_m}{\partial \xi}(x(s), \xi(s))$ to be parallel to $\dot{x}(s)$ since otherwise we could find some $s_0 \in \mathbb{R}$ for which $\operatorname{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_k}(x(s_0))$ was not positive definite on vectors orthogonal to $\dot{x}(s_0)$. After a reparametrisation, we can then assume that

$$\dot{x}(s) = \frac{\partial p_m}{\partial \xi}(x(s), \xi(s))$$

for all $s \in \mathbb{R}$. Plugging this into condition (3.4) yields

$$0 = \frac{\partial p_m}{\partial x_i} + \frac{\mathrm{d}x_k}{\mathrm{d}s} \frac{\partial^2 \psi}{\partial x_k \partial x_i} = \frac{\partial p_m}{\partial x_i} + \frac{\mathrm{d}x_k}{\mathrm{d}s} \frac{\partial \xi_i}{\partial x_k}$$

from which we obtain that

$$\dot{\xi}(s) = -\frac{\partial p_m}{\partial x}(x(s), \xi(s)) \ .$$

Hence, we cannot construct a Gaussian beam along Γ, unless $(x(s), \xi(s))$ is a bicharacteristic curve, cf. definition below.

Definition 3.1 *A curve $(x(s), \eta(s))$ is a bicharacteristic for a linear partial differential operator* $\mathrm{P}(x, \mathrm{D})$ *of order m if it is a solution of*

$$\dot{x}(s) = \frac{\partial p_m}{\partial \eta}(x(s), \eta(s)) \quad \text{and} \quad \dot{\eta}(s) = -\frac{\partial p_m}{\partial x}(x(s), \eta(s)) \ ,$$

where p_m is the principal symbol of $\mathrm{P}(x, \mathrm{D})$.

It follows straight from the definition that $p_m(x(s), \eta(s))$ is constant along any bicharacteristic curve $(x(s), \eta(s))$. This helps in dealing with requirement (3.3) in our Gaussian beam construction. Provided that for the curve Γ along which we want to construct a Gaussian beam the curve

$$(x(s), \xi(s)) = \left(x(s), \frac{\partial \psi}{\partial x}(x(s)) \right)$$

is indeed a bicharacteristic, we only need to check if

$$p_m(x(0), \xi(0)) = 0$$

to ensure that (3.3) holds for all $s \in \mathbb{R}$. Bicharacteristic curves which satisfy (3.3) are commonly referred to as null bicharacteristics.

Hence, so far we have established that unless $(x(s), \xi(s))$ is a null bicharacteristic curve there is no hope of constructing a Gaussian beam along $\Gamma = \{x(s)\}$. Therefore, throughout the remainder of the construction we shall assume that $(x(s), \xi(s))$ is a null bicharacteristic curve. Under this assumption we are also guaranteed that $\dot{x}(s) \neq 0$ for all $s \in \mathbb{R}$ due to the following reasons. Since $\mathrm{P}(x, \mathrm{D})$ is a strictly hyperbolic operator the polynomial

$$g(\xi_0) = p_m((x_0, x_1, \ldots, x_n), (\xi_0, \xi_1, \ldots, \xi_n))$$

cannot have multiple roots for $(\xi_1, \ldots, \xi_n) \neq 0$. In particular, from $p_m(x(s), \xi(s)) \equiv 0$ it follows that

$$\frac{\partial p_m}{\partial \xi_0}(x(s), \xi(s)) \neq 0$$

and therefore,

$$\dot{x}(s) = \frac{\partial p_m}{\partial \xi}(x(s), \xi(s)) \neq 0 \ ,$$

as claimed. Moreover, note that for bicharacteristic curves $(x(s), \xi(s))$ condition (3.4) is only the compatibility condition

$$\dot{\xi}_i(s) = \frac{\partial^2 \psi}{\partial x_i \partial x_j} \dot{x}_j(s) \ .$$

Let us now consider what conditions we obtain by requiring that $f(x) = p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanishes to second order on Γ. As we will see below this gives rise to a non-linear ordinary differential equation and the crucial part of the Gaussian beam construction will be to show that one can solve this differential equation globally. We want

$$\begin{aligned} 0 &= \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= \frac{\partial^2 p_m}{\partial x_j \partial x_i} + \frac{\partial^2 p_m}{\partial \xi_k \partial x_i}\frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial^2 p_m}{\partial x_j \partial \xi_k}\frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_m}{\partial \xi_l \partial \xi_k}\frac{\partial^2 \psi}{\partial x_i \partial x_k}\frac{\partial^2 \psi}{\partial x_j \partial x_l} + \frac{\partial p_m}{\partial \xi_k}\frac{\partial^3 \psi}{\partial x_j \partial x_k \partial x_i} \end{aligned} \tag{3.5}$$

to hold along Γ for $i, j = 0, 1, \ldots, n$. Introducing the matrices

$$(M(s))_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s))\,, \qquad (A(s))_{ij} = \frac{\partial^2 p_m}{\partial x_i \partial x_j}(x(s), \xi(s))\,, \qquad (B(s))_{ij} = \frac{\partial^2 p_m}{\partial \xi_i \partial x_j}(x(s), \xi(s))$$

and

$$(C(s))_{ij} = \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j}(x(s), \xi(s))$$

one can rewrite the second order condition (3.5) as the matrix equation

$$0 = A + MB + B^{\mathrm{T}}M + MCM + \frac{\mathrm{d}M}{\mathrm{d}s}\,. \tag{3.6}$$

Note that the matrices $A(s)$, $B(s)$ and $C(s)$ are known as both the principal symbol p_m and the null bicharacterisic curve $(x(s), \xi(s))$ are given. Thus, in order to have $p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanish to second order on Γ, we need to construct ψ such that the matrix $M(s)$ satisfies the non-linear ordinary differential equation (3.6) globally on Γ. This turns out to be possible due to (3.6) being a Ricatti equation for the matrix $M(s)$.

Moreover, when solving (3.6) for the matrix $M(s)$, we want the solution $M(s)$ to be symmetric, we need

$$M(s)\dot{x}(s) = \dot{\xi}(s)$$

to hold true for all $s \in \mathbb{R}$ and due to the desired condition (b) on ψ we also want $\operatorname{Im} M(s)$ to be positive definite on the orthogonal complement of $\dot{x}(s)$. As we see below, it will be enough to ensure that $M(0)$ has these three properties.

To construct a solution to (3.6), we start by choosing matrix solutions to the linear system

$$\begin{aligned} \dot{Y} &= BY + CN \\ \dot{N} &= -AY - B^{\mathrm{T}}N\,. \end{aligned} \tag{3.7}$$

By linearity, there exists a unique global solution $(Y(s), N(s))$ to this system of ordinary differential equations for any initial data $(Y(0), N(0))$. Furthermore, if $Y(s)$ is invertible around $s = s_0$ then NY^{-1} is a solution to (3.6) around s_0. This follows from the calculation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\left(NY^{-1}\right) &= \dot{N}Y^{-1} - NY^{-1}\dot{Y}Y^{-1} \\ &= -A - B^{\mathrm{T}}NY^{-1} - NY^{-1}B - NY^{-1}CNY^{-1}\,. \end{aligned}$$

The good property of Gaussian beams is that you can choose the initial data $(Y(0), N(0))$ so that $Y(s)$ is invertible for all s. Let $\mathcal{M}$ be a symmetric matrix such that $\operatorname{Im}\mathcal{M}$ is positive definite on the orthogonal complement of $\dot{x}(0)$ and such that $\mathcal{M}\dot{x}(0) = \dot{\xi}(0)$. In the following sequence of lemmas, we establish that if one chooses $(I, \mathcal{M})$ as initial data then $Y(s)$ is invertible for all s and $M(s) = N(s)Y^{-1}(s)$ inherits the three desired properties from $\mathcal{M}$. In particular, note that we have

$$M(0) = N(0)Y^{-1}(0) = \mathcal{M}\,.$$

Lemma 3.2 *Let $(Y(s), N(s))$ be the solution to (3.7) with initial data $(Y(0), N(0)) = (I, \mathcal{M})$. Then it holds true that*

$$(\dot{x}(s), \dot{\xi}(s)) = (Y(s)\dot{x}(0), N(s)\dot{x}(0)) \ .$$

Proof. Using that $(x(s), \xi(s))$ is a bicharacteristic curve one computes

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{x}_i) = \frac{\partial^2 p_m}{\partial x_j \partial \xi_i}\dot{x}_j + \frac{\partial^2 p_m}{\partial \xi_j \partial \xi_i}\dot{\xi}_j = B_{ij}\dot{x}_j + C_{ij}\dot{\xi}_j$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\dot{\xi}_i\right) = -\frac{\partial^2 p_m}{\partial x_j \partial x_i}\dot{x}_j - \frac{\partial^2 p_m}{\partial \xi_j \partial x_i}\dot{\xi}_j = -A_{ij}\dot{x}_j - \left(B^{\mathrm{T}}\right)_{ij}\dot{\xi}_j \ .$$

Thus, $(\dot{x}(s), \dot{\xi}(s))$ and $(Y(s)\dot{x}(0), N(s)\dot{x}(0))$ solve the same linear system of ordinary differential equations. The equality of the two curves follows because we additionally have $Y(0)\dot{x}(0) = \dot{x}(0)$ and

$$N(0)\dot{x}(0) = \mathcal{M}\dot{x}(0) = \dot{\xi}(0)$$

due to our choice of $\mathcal{M}$. □

For the next lemmas, we need the symplectic form $\sigma(\chi_1, \chi_2)$ acting on pairs $\chi_1(s) = (y^1(s), \eta^1(s))$ and $\chi_2(s) = (y^2(s), \eta^2(s))$ of vector solutions to (3.7). The bilinear form $\sigma(\chi_1, \chi_2)$ is given by

$$\sigma(\chi_1, \chi_2) = y^2 \cdot \eta^1 - y^1 \cdot \eta^2 \ .$$

Besides, we need the complexified form

$$\sigma_{\mathbb{C}}(\chi_1, \chi_2) = \sigma(\chi_1, \overline{\chi_2}) \ .$$

Proposition 3.3 *If χ_1 and χ_2 are vector solutions to (3.7) then both the symplectic form $\sigma(\chi_1, \chi_2)$ and the complexified form $\sigma_{\mathbb{C}}(\chi_1, \chi_2)$ are constant in s.*

Proof. The proof relies on the observation that the entries of A, B and C are real and that A and C are symmetric. Differentiating the symplectic form with respect to s yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\sigma(\chi_1, \chi_2) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(y^2 \cdot \eta^1 - y^1 \cdot \eta^2\right) \\ &= \dot{y}^2 \cdot \eta^1 + y^2 \cdot \dot{\eta}^1 - \dot{y}^1 \cdot \eta^2 - y^1 \cdot \dot{\eta}^2 \\ &= By^2 \cdot \eta^1 + C\eta^2 \cdot \eta^1 - y^2 \cdot Ay^1 - y^2 \cdot B^{\mathrm{T}}\eta^1 - By^1 \cdot \eta^2 - C\eta^1 \cdot \eta^2 + y^1 \cdot Ay^2 + y^1 \cdot B^{\mathrm{T}}\eta^2\end{aligned}$$

which equals zero due to

$$y^1 \cdot Ay^2 = \left(y^1\right)^{\mathrm{T}} Ay^2 = \left(y^1\right)^{\mathrm{T}} A^{\mathrm{T}}y^2 = Ay^1 \cdot y^2 \qquad \text{and} \qquad C\eta^2 \cdot \eta^1 = C\eta^1 \cdot \eta^2$$

(using the symmetry of A and C) as well as

$$By^2 \cdot \eta^1 = y^2 \cdot B^{\mathrm{T}}\eta^1 \qquad \text{and} \qquad By^1 \cdot \eta^2 = y^1 \cdot B^{\mathrm{T}}\eta^2 \ .$$

By using that the entries of A, B and C are real, one similarly proves the constancy of the complexified form. □

Lemma 3.4 *Let $(Y(s), N(s))$ be the solution to (3.7) with initial data $(Y(0), N(0)) = (I, \mathcal{M})$. Then $Y(s)$ is invertible for all s.*

Proof. Let s_0 be arbitrary and suppose $Y(s_0)a = 0$ for some vector $a \in \mathbb{C}^{n+1}$. The aim is to deduce that a must then be the zero vector as this will imply that $Y(s_0)$ is invertible.

Let us consider $\chi(s) = (y(s), \eta(s)) = (Y(s)a, N(s)a)$ which has to be a vector solution of (3.7) because $(Y(s), N(s))$ is a solution to (3.7) and a is constant. From the conservation of the complexified form it follows that

$$\begin{aligned} 0 &= \sigma_{\mathbb{C}}(\chi(s_0), \chi(s_0)) = \sigma_{\mathbb{C}}(\chi(0), \chi(0)) = \overline{y(0)} \cdot \eta(0) - y(0) \cdot \overline{\eta(0)} \\ &= \overline{a} \cdot \mathcal{M}a - a \cdot \overline{\mathcal{M}a} \\ &= 2\mathrm{i}\overline{a} \cdot (\operatorname{Im} \mathcal{M})\, a \ . \end{aligned}$$

By assumption, $\operatorname{Im} \mathcal{M}$ is positive definite on the orthogonal complement of $\dot{x}(0)$ and therefore, the last equation implies that $a = \beta\dot{x}(0)$ for some constant $\beta \in \mathbb{C}$. By using Lemma 3.2, we further deduce that

$$0 = Y(s_0)a = \beta Y(s_0)\dot{x}(0) = \beta\dot{x}(s_0) \ .$$

Previously, we have established that $\dot{x}(s_0) \neq 0$ and so it follows that $\beta = 0$. Hence, $a = 0$ and the matrix $Y(s_0)$ is indeed invertible. □

Thus, if $(Y(s), N(s))$ is the solution to (3.7) with initial data $(Y(0), N(0)) = (I, \mathcal{M})$ then

$$M(s) = N(s)Y^{-1}(s)$$

is well-defined and therefore, it is a global solution to (3.6). It remains to prove that $M(s)$ has all the desired properties mentioned above. One of them is an immediate conlusion from Lemma 3.2 as we have

$$\dot{\xi}(s) = N(s)\dot{x}(0) = M(s)Y(s)\dot{x}(0) = M(s)\dot{x}(s)$$

for all $s \in \mathbb{R}$. The other two properties are covered by the following two lemmas.

Lemma 3.5 *Let $(Y(s), N(s))$ be the solution to (3.7) with initial data $(Y(0), N(0)) = (I, \mathcal{M})$. Then $M(s) = N(s)Y^{-1}(s)$ is a symmetric matrix for all $s \in \mathbb{R}$.*

Proof. The proof mainly uses the constancy of the symplectic form. Let $y^i(s)$, $0 \leq i \leq n$ denote the column vectors of $Y(s)$, let $\eta^i(s)$, $0 \leq i \leq n$ denote those of $N(s)$ and let $\chi_i(s) = (y^i(s), \eta^i(s))$. By construction of $M(s)$, we have $\eta^i(s) = M(s)y^i(s)$ and therefore, for any $i, j \in \{0, 1, \ldots, n\}$ it holds true that

$$\sigma(\chi_i(s), \chi_j(s)) = y^j(s) \cdot \eta^i(s) - y^i(s) \cdot \eta^j(s) = y^j(s) \cdot M(s)y^i(s) - y^i(s) \cdot M(s)y^j(s) \ .$$

Due to the symmetry of $M(0) = \mathcal{M}$ and the constancy of the symplectic form, it follows that

$$\begin{aligned} y^j(s) \cdot M(s)y^i(s) - y^i(s) \cdot M(s)y^j(s) &= \sigma(\chi_i(s), \chi_j(s)) \\ &= \sigma(\chi_i(0), \chi_j(0)) = y^j(0) \cdot M(0)y^i(0) - y^i(0) \cdot M(0)y^j(0) = 0 \ . \end{aligned}$$

On the other hand, by Lemma 3.4 we know that the vectors $y^i(s_0)$, $0 \leq i \leq n$ form a basis of $\mathbb{C}^{n+1}$ and hence, the latter equation implies that $M(s)$ is symmetric for all $s \in \mathbb{R}$. □

Lemma 3.6 *Let $M(s)$ be given as in the previous lemma. Then for all $s \in \mathbb{R}$ the matrix $\operatorname{Im} M(s)$ is positive definite on the orthogonal complement of $\dot{x}(s)$*

Proof. This proof relies on the conservation of the complexified form $\sigma_{\mathbb{C}}$. Fix $s_0 \in \mathbb{R}$ and let $y(s_0)$ be an arbitrary vector in the orthogonal complement of $\dot{x}(s_0)$. In particular, this means that $y(s_0)$ is non-zero. Due to Lemma 3.4 there exist $b_0, b_1, \ldots, b_n \in \mathbb{C}$ such that

$$y(s_0) = \sum_{i=0}^{n} b_i y^i(s_0) \ ,$$

where $y^i(s)$, $0 \leq i \leq n$ are the column vectors of $Y(s)$. Similarly, for $\eta^i(s)$, $0 \leq i \leq n$, being the column vectors of $N(s)$, we introduce

$$\chi(s) = \sum_{i=0}^{n} b_i \left(y^i(s), \eta^i(s)\right) \ .$$

As before, one can compute that

$$\sigma_{\mathbb{C}}(\chi(s),\chi(s)) = 2\mathrm{i}\overline{y(s)}\cdot \mathrm{Im}\,(M(s))\,y(s)$$

for $y(s) = \sum_{i=0}^{n} b_i y^i(s)$. Moreover, if $y(0)$ was of the form $y(0) = \beta\dot{x}(0)$ for some constant β then we would get $y(s_0) = \beta\dot{x}(s_0)$ as a consequence of Lemma 3.2. However, this contricts our assumption that $y(s_0)$ lies in the orthogonal complement of $\dot{x}(s_0)$. Thus, the vector $y(0)$ cannot be parallel to $\dot{x}(0)$. Since $M(0) = \mathcal{M}$ is positive definite on orthogonal complement of $\dot{x}(0)$, it follows that

$$\sigma_{\mathbb{C}}(\chi(0),\chi(0)) > 0$$

and hence, by the constancy of the complexified form

$$2\mathrm{i}\overline{y(s_0)}\cdot \mathrm{Im}\,(M(s_0))\,y(s_0) = \sigma_{\mathbb{C}}(\chi(s_0),\chi(s_0)) = \sigma_{\mathbb{C}}(\chi(0),\chi(0)) > 0\ .$$

As $y(s_0)$ was an arbitrary vector orthogonal to $\dot{x}(s_0)$, we deduce that $\mathrm{Im}\,M(s)$ is indeed positive definite on the orthogonal complement of $\dot{x}(s)$ for all $s \in \mathbb{R}$. □

In conclusion, provided that

$$\mathrm{Im}\,\frac{\partial^2\psi}{\partial x_i \partial x_j}(x(0))$$

is positive definite on the orthogonal complement of $\dot{x}(0)$, we can make $p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanish to second order on Γ. This completes the crucial part of the construction of the phase ψ.

By all means, we may want to require that $f(x) = p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanishes on Γ to higher order than two. However, it turns out that this gives rise to linear ordinary differential equations with which it is easier to deal than with the second order condition (3.5). More precisely, for any multi-index α of length r the equations $0 = \partial_x^\alpha f$ along Γ are of the form

$$0 = \frac{\partial p_m}{\partial \xi_j}\frac{\partial}{\partial x_j}\left(\partial_x^\alpha \psi\right) + \sum_{|\beta|=r} c_{\alpha\beta}\partial_x^\beta \psi + d_\alpha\ , \tag{3.8}$$

cf. [2], where the coefficients $c_{\alpha\beta}$ and d_α depend on the partial derivatives up to order $r-1$. Since

$$\frac{\partial p_m}{\partial \xi}\cdot\frac{\partial\left(\partial_x^\alpha\psi\right)}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\partial_x^\alpha\psi\right)$$

we can solve the equations (3.8) as a linear system of ordinary differential equations in s. By linearity, there exists a unique global solution to this system for any initial data. Thus, it suffices to prescribe $\partial_x^\alpha\psi(x(0))$ for $|\alpha| = r$ to get the r^{th} order partial derivatives of ψ on the whole curve Γ which make $p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanish to r^{th} order. We only need to take care of two things. Firstly, due to the dependency of $c_{\alpha\beta}$ and d_α on lower order partial derivatives we need to determine the partials of ψ recursively. Secondly, we need to ensure that they satsify compatibiliy conditions such as

$$\frac{\partial^3\psi}{\partial x_i\partial x_j\partial x_k}(x(s))\dot{x}_k(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}(x(s))\right)\ .$$

However, it suffices to choose the partial derivatives of ψ to be compatible at $x(0)$ as they will then stay compatible for all s.

Overall, we have established that we can make $p_m\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanish to arbitrary finite order on Γ provided that

$$\frac{\partial\psi}{\partial x}(x(s)) = \xi(s)\ ,$$

where $(x(s), \xi(s))$ is the null bicharacteristic curve corresponding to Γ, as well as that

$$\operatorname{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(0))$$

is positive definite on the orthogonal complement of $\dot{x}(0)$. Throughout the remainder of this section, assume that $p_m\left(x, \frac{\partial \psi(x)}{\partial x}\right)$ vanishes to finite order R on Γ.

Having finished the construction of the phase ψ, we still need to determine the Taylor series of $a_0(x), a_1(x), \ldots, a_N(x)$ along the curve Γ. By going back to our ansatz (3.1), we see that the only powers of k which $\mathrm{P}(x, \mathrm{D})u$ can contain are $k^m, k^{m-1}, \ldots, k^{-N+1}, k^{-N}$. Thus, $\mathrm{P}(x, \mathrm{D})u$ is of the form

$$\mathrm{P}(x, \mathrm{D})u = \left(\sum_{s=-m}^{N} c_s(x) k^{-s}\right) \mathrm{e}^{\mathrm{i}k\psi(x)}$$

for coefficients $c_{-m}(x), c_{-m+1}(x), \ldots, c_N(x)$. We already saw that

$$c_{-m}(x) = p_m\left(x, \frac{\partial \psi}{\partial x}\right) a_0(x) \; .$$

To determine $c_{-m+1}(x)$, we need to have a look at how $O(k^{m-1})$ terms occur in $\mathrm{P}(x, \mathrm{D})u$. One such term simply arises from the order $m-1$ terms in $\mathrm{P}(x, \mathrm{D})$, i.e. one contribution to $c_{-m+1}(x)$ is

$$p_{m-1}\left(x, \frac{\partial \psi}{\partial x}\right) a_0(x) \; ,$$

where p_{m-1} is the symbol for the terms of order $m-1$ in $\mathrm{P}(x, \mathrm{D})$. Similar to c_{-m}, another term in c_{-m+1} is given by

$$p_m\left(x, \frac{\partial \psi}{\partial x}\right) a_1(x) \; .$$

The last two terms in c_{-m+1} arise from all the order k terms in $\mathrm{P}(x, \mathrm{D})$ acting on the $a_0(x)\, \mathrm{e}^{\mathrm{i}k\psi(x)}$ part of $u(x, k)$. To get a term of $O(k^{m-1})$ we need $k-1$ of the x-derivatives to act on $\mathrm{e}^{\mathrm{i}k\psi(x)}$ with the other derivative acting on the product of a_0 and terms obtained by differentiating $\mathrm{e}^{\mathrm{i}k\psi(x)}$ with respect to x. In total, one gets

$$\begin{aligned} c_{-m+1}(x) = \frac{1}{\mathrm{i}}\left(\frac{\partial p_m}{\partial \xi_j}\left(x, \frac{\partial \psi}{\partial x}\right) \frac{\partial a_0}{\partial x_j}\right) + \left(\frac{1}{2\mathrm{i}} \frac{\partial^2 p_m}{\partial \xi_j \partial \xi_k}\left(x, \frac{\partial \psi}{\partial x}\right) \frac{\partial^2 \psi}{\partial x_j \partial x_k} + p_{m-1}\left(x, \frac{\partial \psi}{\partial x}\right)\right) a_0 \\ + p_m\left(x, \frac{\partial \psi}{\partial x}\right) a_1 \end{aligned} \tag{3.9}$$

which is of the form

$$c_{-m+1}(x) = \mathrm{L}\, a_0 + p_m\left(x, \frac{\partial \psi}{\partial x}\right) a_1 \; .$$

Similarly, it is possible to show that

$$c_{-m+r+1}(x) = \mathrm{L}\, a_r + p_m\left(x, \frac{\partial \psi}{\partial x}\right) a_{r+1} + g_r \; , \qquad r = 1, \ldots, N+m$$

where g_r is a function depending on $\psi, a_0, \ldots, a_{r-1}$ and their derivatives and where $a_r \equiv 0$ for $r > N$. Thus, determining the partials of $a_0, a_1, \ldots, a_N$ along Γ again reduces to solving linear systems of ordinary differential equations.

From (3.9) we deduce that whenever $p_m\left(x, \frac{\partial \psi}{\partial x}\right)$ vanishes to order R on Γ we can choose the Taylor series of a_0 on Γ up to order $R-2$ in such a way that $c_{-m+1}(x)$ vanishes to order $R-2$ on Γ. Similarly, we can choose the Taylor series of $a_{r-1}(x)$ so that $c_{-m+r}(x)$ vanishes up to order $R-2r$ on Γ.

This nearly concludes the construction of Gaussian beams. As mentioned right at the beginning of the section it only remains to multiply $u(x,k)$ by a k-independent function which vanishes outside a small neighbourhood $\mathscr{O}$ of Γ but which is also identically one on a smaller neighbourhood of Γ. This ensures that $u(x,k)$ really does become concentrated on Γ as $k\to\infty$.

Even though we now finished off the construction of Gaussian beams, we still need to justify that we actually met our original aim (3.2), i.e. we still need to show that $u(x,k)$ is indeed an asymptotic solution to the partial differential equation $\mathrm{P}(x,\mathrm{D})u=0$. For this, we make use of the following lemma.

Lemma 3.7 *Let $T>0$ be given and let $c(x)$ be a function on $\mathbb{R}^{n+1}$ which vanishes to order $S-1$ on the curve Γ, some $S\geq 2$. Suppose both that $\operatorname{supp} c\cap\{|x_0|\leq T\}$ is compact and that $\operatorname{Im}\psi(x)\geq ad^2(x)$ on this set for some constant $a>0$, where $d(x)$ denotes the distance from the point $x\in\mathbb{R}^{n+1}$ to Γ. Then there exists a constant C such that*

$$\int_{|x_0|\leq T}\left|c(x)\,\mathrm{e}^{\mathrm{i}k\psi(x)}\right|^2\,\mathrm{d}x\leq Ck^{-S-n/2}\,.$$

Proof. In a neighbourhood of $\Gamma\cap\{|x_0|\leq T\}$ we can choose k-independent coordinates $z_0,z_1,\ldots,z_n$ such that the curve Γ is parametrised by $z_0=s,z_1=0,\ldots,z_n=0$ and such that we also have

$$d^2(x(z))\geq z_1^2+\cdots+z_n^2\,. \tag{3.10}$$

Using these coordinates, we then introduce the k-dependent coordinates $y_0,y_1,\ldots,y_n$ given by

$$y_0=z_0\,,\qquad y_1=k^{1/2}z_1\,,\qquad\ldots,\qquad y_n=k^{1/2}z_n\,.$$

From (3.10) as well as our assumption on $\operatorname{Im}\psi(x)$ we deduce that

$$\begin{aligned}\left|\exp\left(\mathrm{i}k\psi\left(x\left(y_0,k^{-1/2}y_1,\ldots,k^{-1/2}y_n\right)\right)\right)\right|&\leq\exp\left(-kad^2\left(x\left(y_0,k^{-1/2}y_1,\ldots,k^{-1/2}y_n\right)\right)\right)\\&\leq\exp\left(-a\left(y_1^2+\cdots+y_n^2\right)\right)\leq 1\,.\end{aligned}$$

Finally, we want to change from x to y variables in the integral $\int_{|x_0|\leq T}\left|c(x)\,\mathrm{e}^{\mathrm{i}k\psi(x)}\right|^2\,\mathrm{d}x$. Since the Jacobian of the transformation from x to z coordinates is independent of k whereas the Jacobian for changing variables from z to y is equal to $k^{-n/2}$, it follows that the new integrand is bounded above by

$$\begin{aligned}&Ck^{-n/2}\left|c\left(x\left(y_0,k^{-1/2}y_1,\ldots,k^{-1/2}y_n\right)\right)\right|^2\exp\left(-2a\left(y_1^2+\cdots+y_n^2\right)\right)\\&\qquad\leq Ck^{-S-n/2}\left|k^{S/2}c\left(x\left(y_0,k^{-1/2}y_1,\ldots,k^{-1/2}y_n\right)\right)\right|^2\end{aligned}$$

for some constant C. However, by assumption c vanishes to order $S-1\geq 1$ on Γ and $\operatorname{supp} c\cap\{|x_0|\leq T\}$ is compact. Therefore,

$$\left|k^{S/2}c\left(x\left(y_0,k^{-1/2}y_1,\ldots,k^{-1/2}y_n\right)\right)\right|$$

remains bounded on $\operatorname{supp} c\cap\{|x_0|\leq T\}$ as $k\to\infty$ and the estimate of the lemma follows. □

By a repeated application of Lemma 3.7, we are now able to estimate the Sobolev s-norm $\|\mathrm{P}\,u\|_s$ of $\mathrm{P}(x,\mathrm{D})u$ on $\{|x_0|\leq T\}$. Since $c_{-m+r}(x)$, $r=0,\ldots,N+m$, vanishes to order $R-2r$ on Γ the lemma yields

$$\int_{|x_0|\leq T}\left|c_{-m+r}(x)k^{m-r}\,\mathrm{e}^{\mathrm{i}k\psi(x)}\right|^2\,\mathrm{d}x\leq Ck^{2(m-r)}k^{-(R+1-2r)-n/2}=Ck^{2m-(R+1)-n/2}$$

provided we choose the neighbourhood $\mathscr{O}$ which was introduced above so that $\operatorname{Im}\psi(x)\geq ad^2(x)$ for $x\in\mathscr{O}\cap\{|x_0|\leq T\}$. By further using the inequality $(w_1+w_2+\cdots+w_l)^2\leq 2(w_1^2+w_2^2+\cdots+w_l^2)$, we deduce that

$$\begin{aligned}\|\mathrm{P}\,u\|_0 &= \left(\int_{|x_0|\le T} |\mathrm{P}\,u|^2\,\mathrm{d}x\right)^{1/2} = \left(\int_{|x_0|\le T}\left|\left(\sum_{j=-m}^{N} c_j(x)k^{-j}\right)\mathrm{e}^{\mathrm{i}k\psi(x)}\right|^2\mathrm{d}x\right)^{1/2}\\ &\le \left(2\sum_{j=-m}^{N}\int_{|x_0|\le T}\left|c_j(x)k^{-j}\,\mathrm{e}^{\mathrm{i}k\psi(x)}\right|^2\mathrm{d}x\right)^{1/2}\\ &\le Dk^{m-(R+1)/2-n/4}\,,\end{aligned}$$

for some k-independent constant D. Noting that differentiating $\mathrm{P}(x,\mathrm{D})u$ with respect to x only multiplies the coefficients $c_j(x)$ by k or decreases the order to which they vanish on Γ by at most one, we similarly prove that

$$\|\mathrm{P}\,u\|_s \le Dk^{m+s-(R+1)/2-n/4}\,. \tag{3.11}$$

Thus, if we choose R large enough then we can indeed achieve our aim (3.2).

We end the general discussions with a remark, whose importance will become clear in the propagation of singularities section.

Remark 3.8 *By going back to the Gaussian beam construction, one sees that it is possible to choose the phase ψ depending smoothly on $(x(0),\xi(0))$ as well as on its Taylor series at $x(0)$ up to order R in a way which makes $p_m\left(x,\frac{\partial\psi}{\partial x}\right)$ vanish to order R on $\Gamma\cap\{|x_0|<T\}$.*

Similarly, each $a_{r-1}(x)$ can be chosen as a smooth function of its Taylor series at $x(0)$ to order $R-2r$, the Taylor series of $a_0(x),\dots,a_{r-2}(x)$ at $x(0)$ up to orders $R-2,\dots,R-2(r-1)$, respectively, and the Taylor series of ψ at $x(0)$ to order R.

Finally, one can also establish that the constant C in Lemma 3.7 is uniform both in $(x(0),\xi(0))$ and in the Taylor series of ψ and a_{r-1} up to orders R and $R-2r$, respectively, provided that all the data in consideration lie on a bounded set and that we have a uniform bound on the positive definiteness of

$$\mathrm{Im}\,\frac{\partial^2\psi}{\partial x_i\partial x_j}(x(0))$$

on the orthogonal complement of $\dot{x}(0)$.

3.1 The construction for the wave equation

Having discussed the construction of Gaussian beams for a general strictly hyperbolic partial differential operator $\mathrm{P}(x,\mathrm{D})$ we conclude this section by demonstrating the construction for the two-dimensional wave equation

$$\Box u \equiv -\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}=0\,.$$

To be consistent with the notation used in the first part of this section, we again replace t by x_0. The corresponding principal symbol is then

$$p_2(x,\xi)=-\xi_0^2+\xi_1^2+\xi_2^2\,.$$

First, we note that the wave equation is indeed strictly hyperbolic because for any fixed $(\xi_1,\xi_2)\neq 0$ the polynomial

$$g(\xi_0)=-\xi_0^2+\xi_1^2+\xi_2^2$$

has two distinct real roots, as required. Thus, we can construct Gaussian beams along its null bicharacteristic curves. They are given as solutions of

$$\dot{x}_0=-2\xi_0,\qquad \dot{x}_i=2\xi_i,\quad i=1,2,\qquad\text{and}\qquad \dot{\xi}=0$$

subject to the additional condition $p_2(x(s),\xi(s)) = 0$. From $\dot{\xi} = 0$ we deduce that ξ_0, ξ_1 and ξ_2 need to be constant along any bicharacteristic curve, which further implies that $x(s)$ is of the form

$$x(s) = (-2\xi_0 s, 2\xi_1 s, 2\xi_2 s) \ .$$

For instance, the curve

$$(x(s),\xi(s)) = \left(s, 0, s, -\frac{1}{2}, 0, \frac{1}{2}\right)$$

is a bicharacteristic which is clearly null.

We now restrict our attention to this specific null bicharacteristic curve and construct a Gaussian beam along its projection Γ which is given by $(s, 0, s)$. Let us make the ansatz

$$u(x,k) = \mathrm{e}^{\mathrm{i}k\psi(x)}\, a_0(x) \tag{3.12}$$

and as in the general construction, start by determining conditions on the phase ψ. We have already ensured that $(x(s),\xi(s))$ is a null bicharacteristic curve. Furthermore, we need to satisfy

$$\frac{\partial\psi}{\partial x}(x(s)) = \xi(s) \ , \quad \text{i.e.} \qquad \frac{\partial\psi}{\partial x}(s,0,s) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right) \tag{3.13}$$

as well as condition (3.6) which reduces to

$$0 = MCM + \frac{\mathrm{d}M}{\mathrm{d}s} \qquad \text{for} \qquad C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ .$$

To solve the latter for

$$(M(s))_{ij} = \frac{\partial^2\psi}{\partial x_i \partial x_j}(x(s))$$

we first need to choose an appropriate symmetric matrix $M(0)$. On the one hand, it needs to obey

$$M(0)\dot{x}(0) = \dot{\xi}(0) \ , \quad \text{i.e.} \qquad M(0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and on the other hand, we also want $\operatorname{Im} M(0)$ to be positive definite on the orthogonal complement of $\dot{x}(0) = (1, 0, 1)$, which is spanned by $(0, 1, 0)$ and $(-1, 0, 1)$. It is straightforward to check that

$$M(0) = \begin{pmatrix} b\mathrm{i} & 0 & -b\mathrm{i} \\ 0 & a\mathrm{i} & 0 \\ -b\mathrm{i} & 0 & b\mathrm{i} \end{pmatrix}$$

is an admissible choice provided the constants a and b are both positive. To find the matrix $M(s)$ it then remains to solve the linear system

$$\dot{Y} = CN \ , \qquad \dot{N} = 0$$

with initial data $(Y(0), N(0)) = (I, M(0))$. From $\dot{N} = 0$ we get $N(s) = N(0) = M(0)$. Plugging this into the first differential equation gives $\dot{Y} = CM(0)$ whose solution is

$$Y(s) = Y(0) + sCM(0) = I + sCM(0) \ .$$

Thus, we obtain

$$M(s) = N(s)Y^{-1}(s) = M(0)\left(I + sCM(0)\right)^{-1} \ . \tag{3.14}$$

From the general discussion, we know that $p_2\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ will vanish to at least second order on Γ if we choose the phase ψ such that (3.13) and (3.14) are satisfied. An example of a function ψ with proper 1st and 2nd partials on $(s,0,s)$ is

$$\psi(x_0,x_1,x_2) = \frac{x_2-x_0}{2} + \frac{a^2x_0x_1^2}{1+4a^2x_0^2} + \mathrm{i}\left(\left(\frac{a}{1+4a^2x_0^2}\right)\frac{x_1^2}{2} + \frac{b(x_2-x_0)^2}{2}\right) . \tag{3.15}$$

In fact one can check that with this choice of phase, $p_2\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanishes even to third order on Γ.

We now still need to determine the $a_0(x)$ in our ansatz (3.12). As in the general construction, we use $c_{-2+1}(x) = c_{-1}(x)$ to find a function $a_0(x)$ which works. For $j = 0,1,2$ we compute

$$\frac{\partial u}{\partial x_j} = \mathrm{e}^{\mathrm{i}k\psi}\,\frac{\partial a_0}{\partial x_j} + \frac{\partial \psi}{\partial x_j}\mathrm{i}k\,\mathrm{e}^{\mathrm{i}k\psi}\,a_0$$

and

$$\frac{\partial^2 u}{\partial x_j^2} = \mathrm{e}^{\mathrm{i}k\psi}\,\frac{\partial^2 a_0}{\partial x_j^2} + 2\mathrm{i}k\frac{\partial\psi}{\partial x_j}\frac{\partial a_0}{\partial x_j}\,\mathrm{e}^{\mathrm{i}k\psi} + \mathrm{i}k\frac{\partial^2\psi}{\partial x_j^2}\,\mathrm{e}^{\mathrm{i}k\psi}\,a_0 - k^2\left(\frac{\partial\psi}{\partial x_j}\right)^2\mathrm{e}^{\mathrm{i}k\psi}\,a_0 .$$

Collecting the terms in front of k then yields

$$c_{-1}(x) = \sum_{j=0}^{2}\left(2\mathrm{i}\frac{\partial\psi}{\partial x_j}\frac{\partial a_0}{\partial x_j}\,\mathrm{e}^{\mathrm{i}k\psi} + \mathrm{i}\frac{\partial^2\psi}{\partial x_j^2}\,\mathrm{e}^{\mathrm{i}k\psi}\,a_0\right) .$$

Due to

$$\frac{\partial\psi}{\partial x}(s,0,s) = \left(-\frac{1}{2},0,\frac{1}{2}\right) \qquad \text{as well as} \qquad \frac{\mathrm{d}a_0}{\mathrm{d}s}(s,0,s) = \frac{\partial a_0}{\partial x_0}(s,0,s) + \frac{\partial a_0}{\partial x_2}(s,0,s)$$

it follows that $c_{-1}(x)$ vanishes on $(s,0,s)$ if and only if

$$\frac{\mathrm{d}a_0}{\mathrm{d}s} - (\Box\psi)\,a_0 = 0 \tag{3.16}$$

holds on Γ. Since $\Box\psi = -\mathrm{i}a\,(1+2a\mathrm{i}s)^{-1}$ for the phase ψ as given in (3.15), we can use separation of variables to solve (3.16) for $a_0(x(s))$. If we take $a_0(x(0)) = 1$ this simply yields

$$a_0(x(s)) = (1+2a\mathrm{i}s)^{-1/2} , \tag{3.17}$$

where the branch of the square root is chosen so that we really do have $a_0(x(0)) = 1$. This fixes the function $a_0(x)$ on all of Γ. However, we need $a_0(x)$ to be defined globally or at least on a neighbourhood of Γ. A choice of $a_0(x)$ which is consistent with (3.17) is

$$a_0(x_0,x_1,x_2) = (1+2a\mathrm{i}x_0)^{-1/2} .$$

One can check that this makes $c_{-1}(x)$ vanish to first order on Γ, which is exactly what we need as $p_2\left(x, \frac{\partial\psi(x)}{\partial x}\right)$ vanishes to third order on $(s,0,s)$. Finally, from (3.11) it follows immediately that

$$\|\Box u(x,k)\|_0 = \|\,\mathrm{e}^{\mathrm{i}k\psi(x)}\,a_0(x)\|_0 \le Dk^{2+0-(3+1)/2-2/4} = Dk^{-1/2} = O(k^{-1/2}) .$$

Thus, we found functions $\psi(x)$ and $a_0(x)$ turning (3.12) indeed into an asymptotic solution of the wave equation which becomes concentrated on the curve $(s,0,s)$ as $k\to\infty$.

4 Reflections

We have seen so far that we can construct a Gaussian beam on $\mathbb{R}^{n+1}$ satisfying the smallness criterion

$$\|Pu\|_s \leq Ck^{m+s-(R+1)/2-n/4}. \tag{4.1}$$

In this section we want to consider boundary effects. Let therefore Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary and $D := \mathbb{R} \times \Omega$. For $i = 1, \dots, l$, let B_i be a linear differential operator of order m_i with principal symbol b_i and impose the boundary conditions

$$B_i u = 0 \tag{4.2}$$

on $\partial D = \mathbb{R} \times \partial\Omega$. If a ray path $x(s)$ hits ∂D at $x(s_0)$, then the Gaussian beam shall be reflected at $x(s_0)$ according to the boundary conditions. For the reflected Gaussian beam, we start with the ansatz

$$u = \mathrm{e}^{\mathrm{i}k\psi}\left(a_0 + \ldots + \frac{a_N}{k^N}\right) + \sum_{j=1}^{l} \mathrm{e}^{\mathrm{i}k\psi_j}\left(a_0^j + \ldots + \frac{a_N^j}{k^N}\right), \tag{4.3}$$

where the first summand is given by the construction in the previous section. To carry out the construction for the reflected beam, we need to make two essential assumptions.

Assumption 1 (Non-grazing hypothesis) *Let $\nu = (0, \nu')$ denote the inner unit normal to ∂D at $x(s_0)$. We assume that*

$$t \mapsto p_m(x(s_0), \xi(s_0) + t\nu) \tag{4.4}$$

has m distinct roots in the complex plane.

Note that 0 is always a root because $(x(s_0), \xi(s_0))$ is a bicharacteristic.

Example 4.1 *When dealing with the wave operator $\Box$ it becomes evident why this assumption is called the "non-grazing hypothesis". Letting p_2 denote the corresponding principal symbol and using the notation $\xi = (\xi_0, \xi')$, we have*

$$p_2(x(s_0), \xi(x_0) + t\nu) = -|\xi_0(s_0)|^2 + |\xi'(s_0) + t\nu'|^2 \tag{4.5}$$

$$= p_2(x(s_0), \xi(s_0)) + 2t\xi'(s_0) \cdot \nu' = t(t + 2\xi \cdot \nu) \tag{4.6}$$

since $p_2(x(s), \xi(s)) = 0$ for all s. Thus the assumption is equivalent to $\xi(s_0) \cdot \nu \neq 0$, i. e. $\nu \cdot \dot{x}(s_0) \neq 0$, which does not allow for beams hitting the boundary tangentially.

Remark 4.2 *For all operators P, the non-grazing hypothesis implies*

$$\nu \cdot \dot{x}(s_0) = \nu \cdot \frac{\partial p_m}{\partial \xi}(x(s_0), \xi(s_0)) \neq 0. \tag{4.7}$$

but is not necessarily equivalent to this.

Denote the real roots of (4.4) such that

$$\left[\left(\nu \cdot \frac{\partial p_m}{\partial \xi}\right)\left(\frac{\partial p_m}{\partial \xi_0}\right)\right](x(s_0), \xi(s_0) + \tau\nu) > 0. \tag{4.8}$$

by τ_i, $i = 1, \dots, k_0$. All purely complex roots appear in conjugate pairs. Label all purely complex roots with $\mathrm{Im}\tau_j > 0$ by $j \in \{k_0, \dots, (m-k)/2\}$.

Assumption 2 *Define a matrix b with components*

$$b_{ij} = b_i(x(s_0), \xi(s_0) + \tau_j \nu) \tag{4.9}$$

for $i = 1, \dots, l$, $j = 1, \dots, k_0 + (m-k)/2$. We assume that $\operatorname{rank} b = l$. Moreover, we assume that the number of boundary conditions is $l = k_0 + (m-k)/2$.

Let $1 \leq j \leq k_0$. Since $\frac{\partial p_m}{\partial \xi}(x(s_0), \xi(s_0) + \tau_j \nu) \in \mathbb{R}$, we can define the bicharacteristic curve

$$\Gamma_j : s \mapsto (\underline{x}(s), \underline{\xi}(s)) \tag{4.10}$$

starting at $(x(s_0), \xi(s_0) + \tau_j \nu)$ via

$$\underline{\dot{x}} = \frac{\partial p_m}{\partial \xi}, \quad \underline{\dot{\xi}} = -\frac{\partial p_m}{\partial x}. \tag{4.11}$$

Lemma 4.3 *The ray path Γ_j moves forward in time, i. e. $x_0 > x_0(s_0)$ on Γ_j in D.*

Proof. In terms of the bicharacteristics defined above, equation (4.8) reads as

$$(\nu \cdot \underline{\dot{x}}(s_0))\underline{\dot{x}_0}(s_0) > 0. \tag{4.12}$$

Thus Γ_j enters D as $\underline{x_0}$ increases. Strict hyperbolicity implies that

$$\underline{\dot{x}_0}(s) = \frac{\partial p_m}{\partial \xi_0}(\underline{x}(s), \underline{\xi}(s)) \neq 0 \tag{4.13}$$

and thus $x_0 > x_0(s_0)$ on Γ_j in D. □

Remark 4.4 *Note that if $\underline{x_0}(s)$ decreases as s increases, we follow the ray path backwards in s.*

Lemma 4.5 *Assume $1 \leq j \leq k_0$. Let $\psi = \psi_j$ on ∂D and*

$$\frac{\partial \psi_j}{\partial x}(\underline{x}(s)) = \underline{\xi}(s). \tag{4.14}$$

Then the function ψ_j can be chosen so that $p_m(x, \partial\psi_j/\partial x)$ vanishes to order R on the curve Γ_j with

$$\left(\mathrm{Im} \frac{\partial \psi_j}{\partial x_i \partial x_k}(\underline{x}(s))\right)_{i,k} \tag{4.15}$$

positive definite on the orthogonal complement of $\underline{\dot{x}}(s)$.

Remark 4.6 *Note that our requirement on $\partial\psi_j/\partial x$ is compatible with $\psi|_{\partial D} = \psi_j|_{\partial D}$ since $\underline{\xi}(s_0) = \xi(s_0) + \tau_j \nu$.*

Proof. First observe that the compatibility condition

$$\sum_{i=0}^{n} \frac{\partial^{|\alpha|+1} \psi_j}{\partial x^\alpha \partial x_i}(x(s_0)) \frac{\partial p_m}{\partial \xi_i}(x(s_0), \xi(s_0) + \tau_j \nu) = \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^{|\alpha|} \psi_j}{\partial x^\alpha}(\underline{x}(s))|_{s=0} \tag{4.16}$$

holds for the αth derivative of ψ_j at $x(s_0)$. In particular, this implies that $\psi(\underline{x}(s)) \in \mathbb{R}$ by (4.14). As

$$\left(\mathrm{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_k}(x(s))\right)_{i,k} \tag{4.17}$$

is positive definite on vectors orthogonal to $\dot{x}(s)$, but zero tangential to Γ, we find that

$$\left(\mathrm{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_k}(x(s_0))\right)_{i,k} \tag{4.18}$$

is positive definite on the tangential plane to ∂D on $x(s_0)$ which is not tangential to the curve by (4.7). Since $\psi_j = \psi$ on ∂D,

$$\left(\mathrm{Im} \frac{\partial^2 \psi_j}{\partial x_i \partial x_k}(x(s_0))\right)_{i,k} \tag{4.19}$$

is also positive definite on this plane. We have already established that $\psi(\underline{x}(s)) \in \mathbb{R}$ for all s, thus $\left(\mathrm{Im}(\partial^2\psi_j/\partial x_i \partial x_k)x(s_0)\right)_{i,k} = 0$ on vectors parllel to $\underline{\dot{x}}$. Therefore we conclude that

$$\left(\mathrm{Im}\frac{\partial^2 \psi_j}{\partial x_i \partial x_k}(x(s_0))\right)_{i,k} \tag{4.20}$$

is positive definite on the plane orthogonal to Γ_j by the non-grazing condition

$$\nu \cdot \underline{\dot{x}}(s_0) = \nu \cdot \frac{\partial p_m}{\partial \xi}(x(s_0), \xi(s_0) + \tau_j \nu) = \frac{\mathrm{d} p_m}{\mathrm{d}t}(x(s_0), \xi(s_0) + \tau_j \nu) \neq 0. \tag{4.21}$$

Now it is possible to perform the same construction we used to find ψ in the previous section to construct ψ_j such that

$$p_m\left(x, \frac{\partial \psi_j}{\partial x}\right) = 0 \tag{4.22}$$

vanishes to order R on Γ_j. Since (4.21) holds, equation (4.16) enables us to always express a derivative in ν direction by other derivatives. This was not an issue in the original construction since we started the Gaussian beam at the time slice $\{x_0 = 0\}$ so that the analogue of (4.21) was given by $x_0(s_0) \neq 0$. □

To determine ψ_j for $k_0 + 1 \leq j \leq k_0 + (m-k)/2$ we can exploit a more direct approach.

Lemma 4.7 *Let $j > k_0$ and assume $\psi_j = \psi$ on ∂D with*

$$\frac{\partial \psi_j}{\partial x}(x(s_0)) = \xi(s_0) + \tau_j \nu. \tag{4.23}$$

Then we can construct ψ_j such that $p_m(x, \partial\psi_j/\partial x) = 0$ to order R. Moreover the Taylor series of ψ_j around $x(s_0)$ is determined uniquely and there is a $c > 0$ and a neighbourhood $U \subseteq D$ of $x(s_0)$ such that

$$\mathrm{Im}\psi_j(x) \geq c|x - x(s_0)|^2 \tag{4.24}$$

for all $x \in U$.

Proof. If $p_m(x, \partial\psi_j/\partial x) = 0$ up to order R and $|\alpha| \leq R$, we have the expression

$$0 = \sum_{i=0}^{n} \frac{\partial^{|\alpha|+1}\psi_j}{\partial x^\alpha \partial x_i}(x(s_0)) \frac{\partial p_m}{\partial \xi_i}(x(s_0), \frac{\partial \psi_j}{\partial x}(x(s_0))) + q_\alpha(x(s_0)), \tag{4.25}$$

where q_α is a function depending only on $\partial^{|\beta|}\psi_j/\partial x^\beta$ for $|\beta| \leq |\alpha|$.

Since $\psi_j = \psi$ on the boundary and

$$\nu \cdot \frac{\partial p_m}{\partial \xi}(x(s_0), \xi(s_0) + \tau_j \nu) \neq 0 \tag{4.26}$$

by the non-grazing condition, this determines the Taylor series uniquely.

For the rest of the proof, assume that near $x(s_0)$ the domain D is locally defined by x_n and $x(s_0) = 0$. Then

$$\mathrm{Im}\frac{\partial \psi_j}{\partial x}(x(s_0)) = (0, \ldots, 0, \mathrm{Im}\tau_j) \tag{4.27}$$

and

$$\left(\mathrm{Im}\frac{\partial^2 \psi_j}{\partial x_i \partial x_k}(x(s_0))\right)_{i,k} = \left(\mathrm{Im}\frac{\partial^2 \psi}{\partial x_i \partial x_k}(x(s_0))\right)_{i,k} =: A \tag{4.28}$$

for $0 \le i,k \le n-1$. Using the notation $x = (x', x_n)$, this yields

$$\mathrm{Im}\psi_j = (\mathrm{Im}\tau_j)x_n + \frac{1}{2}x' \cdot Ax' + x_n K \cdot x' + O(x_n^2) + O(|x - x(s_0)|^3). \tag{4.29}$$

Since A is positive definite and $\mathrm{Im}\tau_j > 0$, $x_n(\mathrm{Im}\tau_j - K) > 0$ for $|x'| < \mathrm{Im}\tau_j/|K|$. Hence, for $|x - x(s_0)|$ small enough, there is a constant $c > 0$ such that $\mathrm{Im}\psi_j(x)$ satisfies the inequality stated above. □

It remains to construct the coefficients $a_0^j, \ldots, a_N^j$. For u satisfying the ansatz (4.3), we will write

$$Pu = \sum_{r=-m}^{N} \left(c_r k^{-r} \, \mathrm{e}^{\mathrm{i}k\psi} + \sum_{j=1}^{l} c_r^j k^{-r} \, \mathrm{e}^{\mathrm{i}k\psi_j} \right). \tag{4.30}$$

Since $\psi = \psi_j$ on ∂D, we get

$$B_j u = \sum_{r=-m_j}^{N} d_r^j(x) k^{-r} \, \mathrm{e}^{\mathrm{i}k\psi}. \tag{4.31}$$

Lemma 4.8 *One can choose the Taylor series of $a_0^j, \ldots, a_N^j$ such that such that $d_{-m_j+s}^j$ vanishes to order $R-2s$ on ∂D and $c_{-m+s}^j(x)$ vanishes on Γ_j to order $R-2s$ for $j \le k_0$ and $c_{-m+s}^j(x)$ vanishes to order $R-2s$ at $x(s_0)$ for $j > k_0$.*

Proof. Using the ansatz (4.3) and assuming that B_j is of order m_j, we get

$$B_j u = \sum_{r=-m_j}^{N} d_r^j(x) k^{-r} \, \mathrm{e}^{\mathrm{i}k\psi} \tag{4.32}$$

for $j = 1, \ldots, l$ on ∂D with

$$d_{-m_j}^j = b_j\left(x, \frac{\partial\psi}{\partial x}\right) a_0 + b_j\left(x, \frac{\partial\psi_1}{\partial x}\right) a_0^1 + \ldots + b_j\left(x, \frac{\partial\psi_l}{\partial x}\right) x_0^l \tag{4.33}$$

$$= b_j\left(x, \frac{\partial\psi}{\partial x}\right) a_0 + \sum_{i=1}^{l} b_{ji} a_0^i \tag{4.34}$$

and, for $s \ge 1$,

$$d_{-m_j+s}^j = b_j\left(x, \frac{\partial\psi}{\partial x}\right) a_s + b_j\left(x, \frac{\partial\psi_1}{\partial x}\right) a_s^1 + \ldots + b_j\left(x, \frac{\partial\psi_l}{\partial x}\right) a_s^l + g_{js} \tag{4.35}$$

$$= b_j\left(x, \frac{\partial\psi}{\partial x}\right) a_s + \sum_{i=1}^{l} b_{ji} a_s^i + g_{js}, \tag{4.36}$$

where g_{js} is a function of $a_r, a_r^1, \ldots, a_r^l$ for $r = 0, \ldots, s-1$ and their derivatives. Since b is invertible, $d_{-m_j+s}^j$ vanishing to order $R-2s$ determines the Taylor series of a_s^j uniquely up to order $R-2s$.

For $1 \le j \le k_0$, we can use the method described in the previous section to chose the Taylor series of $a_0^j, \ldots, a_N^j$ on Γ_j such that c_{-m+s}^j vanishes to order $R-2s$ on Γ_j.

For $j > k_0$, we use the analogue for $c_{-m_j+s}^j$ of equation (4.25) ot show that $c_{-m_j+s}^j$ vanishes to order $R-2s$ at $x(s_0)$. □

As in the previous construction, the approximation is finished once we multiply the terms by suitable bump functions in order to localise the solutions in space.

Theorem 4.9 *The above construction gives*

$$\|Pu\|_s \le Ck^{m+s-(R+1)/2-n/4} \tag{4.37}$$

and

$$|||B_i u|||_{s'} \leq C' k^{m_j + s' - (R+1)/2 - n/4} \tag{4.38}$$

where $|||\cdot|||_{s'}$ *denotes the Sobolev norm of order* s' *on* $\partial D \cap \{|x_0| < T\}$.

Proof. One can modify Lemma 3.7 to see that the contributions of

$$\left\| P \, \mathrm{e}^{\mathrm{i}\psi_j} \left(a_1^j + \ldots + \frac{a_N^j}{k^N} \right) \right\|_s \tag{4.39}$$

for $l > k_0$ are of higher order in $1/k$ than than for $l \leq k_0$. For the latter (and for $|||B_i u|||_{s'}$) we can use the results from the previous section. □

Remark 4.10 *The constants in the inequalities are uniform for* $y = x(0)$ *and* $\eta = \xi(0)$ *in a neighbourhood of an admissible value as in the previous section, although now this neighbourhood might have to be small to avoid grazing.*

Remark 4.11 *By Lemma 4.3 all ray paths go forward in time and, since we multiplied with suitable bump functions, none of these modifications made by us in this chapter to account for the boundary conditions will influence the initial values at* $x_0 = 0$.

5 Propagation of Singularities

One of the main differences between hyperbolic and, say elliptic, partial differential equations is (assuming all differential operators have infinitely differentiable coefficients) that they may admit solutions which fail to be smooth. The notion of *propagation of singularities* arises when one studies that points at which solutions of hyperbolic PDEs are not infinitely differentiable. The term propagation refers to the way in which singularities at a given time t_0 translate to singularities at later times.

The Gaussian beam construction for strictly hyperbolic partial differential operators provides asymptotic solutions to the related PDEs which are concentrated near *ray paths*, or projections of null bicharacteristic curves. It turns out that one can use the construction, for an operator P, to prove that singularities in solutions of $Pu = f$ can only propagate along these paths. To demonstrate this, consider the operator $Pu = \frac{\partial u}{\partial t}$: the solutions of $Pu = 0$ are the functions u which are independent of t. This means that there exist solutions which vanish away from lines $\{(x_0, t),\, t \in \mathbb{R}\}$ for fixed $x_0 \in \mathbb{R}^n$, and clearly in this case the singularities of the solutions propagate along these lines.

The idea of a singularity of a distribution can be refined by studying what is known as its *wave front set*, a concept introduced by Hörmander in [1]. As is now standard we will give our propagation of singularities result in terms of wave front sets, and so the following section will consist of a brief introduction to the idea and underlying theory.

5.1 Wave Front Sets

5.1.1 Fourier Transforms

The key motivation for the definitions that follow, is that for an integrable function on $\mathbb{R}^n$ we have a relationship between smoothness, and decay of the Fourier transform at infinity. This essentially arises from the fact that for any f which is sufficiently nice, we have $\widehat{(D_x^\alpha f)}(\xi) = \xi^\alpha \hat{f}(\xi)$ and $\widehat{(x^\beta f)}(\xi) = (-D_\lambda)^\beta \hat{f}(\xi)$ where α and β are multi-indices, and for computational ease we write $D^\alpha = -\mathrm{i}\partial^\alpha$. In fact, as we will see shortly, we can characterize smoothness of a function f in terms of bounds on $\hat{f}$. This characterization can be extended to distributions $f \in \mathcal{D}'(\mathbb{R}^n)$, and motivates the definition of the wave front set in terms of bounds on the Fourier transform.

More precisely, given $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x.\xi} f(x)\, dx$$

and have the folowing results:

Lemma 5.1 *If $f \in C_0^\infty(\mathbb{R}^n)$ then for each $N \in \mathbb{N}$ there is a C_N such that*

$$|\widehat{f}(\xi)| \le C_N (1 + |\xi|)^{-N}. \tag{5.1}$$

Proof. This follows from the fact that if $f \in C_0^\infty(\mathbb{R}^n)$ then

$$(1 + |\xi|^2)^M \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x.\xi} (1 - \Delta)^M f(x)\, dx.$$

which holds since

$$\begin{aligned}
\widehat{((1-\Delta)^M f)}(\xi) &= \int \mathrm{e}^{-\mathrm{i}x.\xi} (1-\Delta)^M f(x)\, dx &(5.2)\\
&= \int_{\mathbb{R}^n} f(x) (1-\Delta)^M \mathrm{e}^{-\mathrm{i}x.\xi}\, dx &(5.3)\\
&= (1+|\xi|^2)^M \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-\mathrm{i}x.\xi}\, dx &(5.4)\\
&= (1+|\xi|^2)^M \widehat{f}(\xi) &(5.5)
\end{aligned}$$

and the 2nd line follows by integration by parts, since the boundary terms disappear for f compactly supported.

From here, writing

$$\widehat{f}(\xi) = (1+|\xi|^2)^{-2M}(1+|\xi|^2)^{2M}\widehat{f}(\xi) = (1+|\xi|^2)^{-2M}((1-\Delta)^M f)\widehat{}(\xi)$$

yields the result, since $f \in C_0^1(\mathbb{R}^n)$ implies that $(1-\Delta)^M f \in L^1$ and so $((1-\Delta)^M f)\widehat{}(\xi)$ is bounded by a constant depending only on M. ☐

Lemma 5.2 *If $f, \widehat{f} \in L^1$ and $\widehat{f}$ obeys the estimates (5.1) then $f(x) \in C^\infty(\mathbb{R}^n)$.*

Proof. Since $f, \widehat{f}$ are both in L^1 the Fourier inversion formula cite[Theorem 9.11]rudin

$$f(x) = (2\pi)^{n/2} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x.\xi}\, \widehat{f}(\xi)\, d\xi \quad \text{a.e.}$$

holds. Then for any multi-index β

$$\begin{aligned}(-D_x)^\beta f(x) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-D_x)^\beta\, \mathrm{e}^{-\mathrm{i}x.\xi}\, \widehat{f}(\xi)\, d\xi && (5.6)\\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} x^\beta \widehat{f}(\xi)\, \mathrm{e}^{-\mathrm{i}x.\xi}, d\xi && (5.7)\\ &= (2\pi)^{-n/2} (x^\beta \widehat{f})\widehat{}(\xi) && (5.8)\end{aligned}$$

where the estimates on $\widehat{f}$ justify differentiating under the integral in line 1. The final expression is continuous as $x^\beta \hat{f} \in L^1$ (another consequence of the bounds (5.1)) and thus has continuous Fourier transform. We have shown that all derivatives of f are continuous, and so have $f \in C^\infty(\mathbb{R}^n)$ as required. ☐

Theorem 5.3 (Characterisation of Smoothness) *A function $f \in L^1_{loc}(\mathbb{R}^n)$ is equivalent to a smooth function in a neighbourhood of x_0 if and only if there is a non-negative function $\rho \in C_0^\infty(\mathbb{R}^n)$ with $\rho(x_0) = 1$ such that*

$$\widehat{\rho f}(\xi)$$

satsifies the estimates (5.1).

Proof. Assume that there exists such a ρ. Then $\rho f \in L^1$ (since $f \in L^1_{\mathrm{loc}}$) and $\widehat{\rho f} \in L^1$ also, as a result of the estimates (5.1). By Lemma 5.2 this means that $\rho f \in C^\infty(\mathbb{R}^n)$ and so f is equivalent to a C^∞ function on a neighborhood of x_0.

Conversely, if f is equivalent to a C^∞ function on a neighborhood of x_0 then building ρ non-negative and smooth, supported on this neighbourhood (wlog assume it is compact) such that $\rho(x_0) = 1$ gives that $\rho f \in C_0^\infty(\mathbb{R}^n)$. Then $\widehat{\rho f}$ satisfies the estimates (5.1) by previous work. ☐

As mentioned before, this characterization can be extended to distributions $f \in \mathcal{D}'(\mathbb{R}^n)$.

Say that $f \in \mathcal{D}'(\mathbb{R}^n)$ is equivalent to a C^∞ function g on a neighbourhood $\mathcal{O} \subset \mathbb{R}^n$ if for all $\varphi \in \mathcal{D}(\mathcal{O})$

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} g\varphi\, dx$$

and for $\rho \in C_0^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ define

$$\widehat{\rho f}(\xi) = \left\langle f, \rho\, \mathrm{e}^{-\mathrm{i}x.\xi} \right\rangle. \tag{5.9}$$

Note that if $f \in \mathcal{D}(\mathbb{R}^n)$ this corresponds to the Fourier transform of ρf as before. With this set up, we obtain the corresponding theorem:

Theorem 5.4 *$f \in \mathcal{D}'(\mathbb{R}^n)$ is equivalent to a C^∞ function on a neighbourhood of x_0 if and only if there exists ρ as before, with $\widehat{\rho f}(\xi)$ satisfying the estimates (5.1).*

Proof. Suppose such a ρ exists, then $\langle f^\rho, \varphi\rangle = \langle f, \rho\varphi\rangle$ defines an element f^ρ of $\mathcal{E}'(\mathbb{R}^n)$ (since ρ is compactly supported.) Its Fourier transform $\widehat{f^\rho} \in \mathcal{S}'(\mathbb{R}^n)$ is in fact a function satisfying

$$\widehat{f^\rho}(\xi) = \left\langle f^\rho, \mathrm{e}^{-\mathrm{i}x.\xi}\right\rangle \;\; \forall \xi$$

since we have, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\left\langle \widehat{f^\rho}, \varphi \right\rangle &= \langle f^\rho, \widehat{\varphi}\rangle && (5.10)\\
&= \left\langle f^\rho, \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x.\xi}\, \varphi(\xi)\, d\xi \right\rangle && (5.11)\\
&= \int_{\mathbb{R}^n} \left\langle f^\rho, \mathrm{e}^{-\mathrm{i}x.\xi}\right\rangle \varphi(\xi)\, d\xi && (5.12)
\end{aligned}$$

where the final line is justified by the fact that f^ρ is compactly supported. Note that this, along with definition (5.9), means that

$$\widehat{f^\rho}(\xi) = \widehat{\rho f}(\xi)$$

for all ξ.

Now, since $f^\rho \in \mathcal{S}'(\mathbb{R}^n)$ ($\mathcal{E}'(\mathbb{R}^n)$ is a subset of this space) the Fourier inversion formula for $\mathcal{S}'(\mathbb{R}^n)$ holds, that is (see [4])

$$(2\pi)^n \check{f}^\rho = \widehat{\widehat{f^\rho}}$$

where for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\check{\varphi}(x) = \varphi(-x)$ and for $g \in \mathcal{D}'(\mathbb{R}^n)$, $\check{g}$ is defined by $\langle \check{g}, \varphi\rangle = \langle g, \check{\varphi}\rangle$.

The Fourier transform of $\widehat{f^\rho}$ must satisfy, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\left\langle \widehat{\widehat{f^\rho}}, \varphi \right\rangle &= \left\langle \widehat{f^\rho}, \hat{\varphi} \right\rangle && (5.13)\\
&= \int_{\mathbb{R}^n} \widehat{f^\rho}(\xi)\widehat{\varphi}(\xi)\, d\xi && (5.14)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{f^\rho}(\xi)\, \mathrm{e}^{-\mathrm{i}x.\xi}\, \varphi(x)\, dx d\xi && (5.15)\\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{f^\rho}(\xi)\, \mathrm{e}^{-\mathrm{i}x.\xi}\;\, d\xi\right) \varphi(x)\, dx && (5.16)
\end{aligned}$$

where the final line uses Fubini's theorem, justified by the bounds (5.1) on $\widehat{\rho f}(\xi) = \widehat{f^\rho}(\xi)$.

The same bounds give that $\int_{\mathbb{R}^n} \widehat{f^\rho}(\xi)\, \mathrm{e}^{-\mathrm{i}x.\xi}\; d\xi$ is a C^∞ function of x (as in the proof of Lemma 5.2) and so $\widehat{\widehat{f^\rho}} = (2\pi)^n \check{f}^\rho$, satisfies

$$\left\langle (2\pi)^n \check{f}^\rho, \varphi \right\rangle = \int_{\mathbb{R}^n} g\varphi\, dx$$

for some C^∞ function g, and all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Clearly this implies we have the same result for f^ρ, with corresponding smooth function g'.

Finally, since $\langle f^\rho, \varphi\rangle = \langle f, \rho\varphi\rangle$ for all φ, we can deduce the existence of a function $\tilde{g}$ such that for all φ supported in $\mathcal{O} \subset \mathrm{supp}(\rho)$, a neighborhood of x_0,

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^n} \tilde{g}\varphi\, dx$$

ie. f is equivalent to a C^∞ function in this neighbourhood.

Conversely, assume f is equivalent to a C^∞ function on $\mathcal{O}$ a neighborhood of x_0, so there exists some function $g \in C^\infty(\mathbb{R}^n)$ such that

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} g\varphi$$

for all $\varphi \in \mathcal{D}(\mathcal{O})$. Construct $\rho \in \mathcal{D}(\mathbb{R}^n)$ non-negative with $\rho(x_0) = 1$ and $\operatorname{supp}(\rho) \subset \mathcal{O}$. Then

$$\widehat{\rho f}(\xi) = \left\langle f, \rho\, \mathrm{e}^{-\mathrm{i}x.\xi} \right\rangle \tag{5.17}$$

$$= \int_{\mathbb{R}^n} g(x)\rho(x)\, \mathrm{e}^{-\mathrm{i}x.\xi}\, dx \tag{5.18}$$

$$= (g\rho)\widehat{\ }(\xi) \tag{5.19}$$

where $g\rho \in C_0^\infty(\mathbb{R}^n)$ since both are smooth and ρ is compactly supported. Hence, by previous work, $\widehat{\rho f}(\xi) = (g\rho)\widehat{\ }(\xi)$ satisfies (5.1) as required. □

5.1.2 Definition

With this characterisation of smoothness in mind, we are ready to define the wave front set of a distribution on $\mathbb{R}^n$. Recall that the *singular support* of a distribution is defined by

$$\operatorname{sing\,supp}(f) = \mathbb{R}^n \setminus \bigcup_{\substack{\mathcal{O}\text{ open} \\ f \equiv g \text{ smooth on } \mathcal{O}}} \mathcal{O}.$$

This is the complement of the largest open set on which f is equivalent to a smooth function, so contains all the singularities of f. The wave front set of f, $WF(f)$, is a refinement of the singular support in the sense that it tells us not only where f fails to be smooth but also in which directions.

Definition 5.5 *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. Say $(x_0, \xi_0) \notin WF(f)$ if and only if there exists $\rho \in C_0^\infty(\mathbb{R}^n)$ with $\rho(x_0) = 1$ and a conic neighbourhood N of ξ_0 such that $\widehat{\rho f}(\xi)$ satistfies (5.1) for all $\xi \in N$.*

Remark 5.6 *We say an open set N is* conic *if $\xi \in N \Rightarrow t\xi \in N$ for all $t > 0$.*

Note that that Theorem 5.4 tells us that if $x_0 \in \mathbb{R}^n$ is not in the singular support of f, then f is smooth in a neighbourhood of x_0 and there must exist $\rho \in C_0^\infty(\mathbb{R}^n)$ non-negative with $\rho(x_0) = 1$ such that $\widehat{\rho f}(\xi)$ satisfies the estimates (5.1) for all ξ. Thus for every $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ we can find a conic neighbourhood N of ξ_0 such that $\widehat{\rho f}(\xi)$ satisfies the estimates whenever $\xi \in N$. This gives us that $(x_0, \xi_0) \notin WF(f)$ for all $\xi_0 \in \mathbb{R}^n \setminus \{0\}$.

Conversely, if this statement holds at x_0, then in particular for each $\xi \in S^{n-1}$ there exists some ρ_ξ satisfying the usual properties, and N_ξ a conic neighbourhood of ξ, such that $|\widehat{\rho_\xi f}(\eta)| \leq C_N(1 + |\eta|)^{-N}$ for all $\eta \in N_\xi$. The neighbourhoods N_ξ, for $\xi \in S^{n-1}$, restrict to open neighbourhoods on the sphere and thus form an open cover of S^{n-1}. By compactness, there must exist a finite subcover $\{\tilde{N}_1, \cdots \tilde{N}_k\}$ of S^{n-1}, where these sets correspond to conic neighbourhoods $\{N_1, \cdots N_k\}$ of points $\{\xi_1, \cdots \xi_k\}$ respectively. Note that for each pair (x_i, N_i) we also have $\rho_i \in C_0^\infty(\mathbb{R}^n)$ such that $\widehat{\rho_i f}(\xi)$ satisfies (5.1) whenever $\xi \in N_i$. Since all the ρ_i's are smooth, compactly supported, non-negative and equal to 1 at x_0, the function $\rho = \rho_1 \times \cdots \times \rho_k$ also satisfies these properties. Moreover, for every $\xi \in \mathbb{R}^n \setminus \{0\}$, we have that $\frac{\xi}{|\xi|} \in S^{n-1}$ and so $\frac{\xi}{|\xi|} \in \tilde{N}_i$ for some $i \in [1, \cdots, k]$. In particular, $\frac{\xi}{|\xi|} \in N_i$ and since each N_i is conic, it must be that $\xi \in N_i$ also.

Since $\left|\widehat{\rho_i f \eta}\right| \leq C_N \left(1+|\eta|\right)^{-N}$ for all $\eta \in N_i$, a conic neighbourhood of ξ, there must also exist a further conic neighbourhood N of ξ such that the same estimates hold for $\left\{\widehat{\rho f}(\eta) \,:\, \eta \in N\right\}$. This is true because $\rho = \psi_i \rho_i$ with $\psi_i = \prod_{j\neq i} \rho_j$ smooth, and from here we can apply an argument that will be detailed shortly in the proof of Theorem 5.9. So for arbitrary $\xi \in \mathbb{R}^n \setminus \{0\}$ we have a conic neighbourhood N'_ξ of ξ such that $\widehat{\rho f}(\eta)$ satisfies estimates of the form (5.1) for $\eta \in N'_\xi$. Though the constants in these estimates may depend on ξ, we can apply the same trick as above (finding a finite subcover of S^{n-1}) to obtain global constants C_N such that $\widehat{\rho f}(\xi) \leq C_N(1+|\xi|)^{-N}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence, by our characterisation of smoothness (Theorem 5.4) it follows that f is smooth in a neighbourhood of x_0, ie. $x_0 \notin \text{singsupp}(f)$.

Putting this together we obtain:

Lemma 5.7

$$x \in \textit{singsupp}(f) \Leftrightarrow (x,\xi) \in WF(f) \textit{ for some } \xi \in \mathbb{R}^n \setminus \{0\}$$

so the wave front set is indeed a refinement of the singular support.

Remark 5.8 *In the discussion of the previous section, the bounds $|\widehat{f}(\xi)| \leq C_N(1+|\xi|)^{-N}$ were only used to deal with the behaviour of the Fourier transform for large values of $|\xi|$. Therefore, we could equivalently replace these with bounds of the form*

$$|\widehat{f}(\xi)| \leq C_N|\xi|^{-N} \;\; \textit{for} \, |\xi| \geq 1. \tag{5.20}$$

Indeed we will use this type of estimate in the sequel, to determine whether or not certain points lie in the wave front set of a given distribution.

To justify this formally, note that for $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\rho \in C_0^\infty(\mathbb{R}^n)$, $|\widehat{\rho f}(\xi)|$ is uniformly bounded in ξ, and so since

$$(1+|\xi|)^{-N} \geq 2^{-N} \, \textit{for} \, |\xi| \leq 1 \;\; \textit{and}$$

$$\left(\frac{|\xi|}{1+|\xi|}\right)^{-N} \leq 2^N \, \textit{for} \, |\xi| \geq 1,$$

given the estimates $\left(\left|\widehat{\rho f}(\xi)\right| \leq C_N|\xi|^{-N}\right)$ for $|\xi| \geq 1$ we may deduce estimates of the form (5.1).

5.1.3 Examples

In this section, some examples of wave front sets will be presented, in order to give a more intuitive idea of what they are and how they can be determined.

To begin, we'll consider a canonical example: the dirac delta function δ_0. Lemma 5.7 tells us immediately that $(x,\xi) \notin WF(\delta_0)$ unless $x = \underline{0}$, since δ_0 is clearly equivalent to a smooth function in the rest of $\mathbb{R}^n$, and so we need only focus on this point. By symmetry, it is natural to think of δ_0 as being singular in all directions at the origin, and indeed the wave front set reflects this. We have:

$$WF(\delta_0) = \{(0,\xi) : \xi \neq 0\}.$$

One can verify this statement easily, since for all $\rho \in C_0^\infty(\mathbb{R}^n)$ with $\rho(0) = 1$

$$\widehat{\rho\delta_0}(\xi) = \left\langle \delta_0, \rho \, \mathrm{e}^{-\mathrm{i}x.\xi} \right\rangle = \rho(0) = 1$$

which does not decay in any direction.

A slightly more interesting example, when directions are clearly distinguished, is the distribution $u = \delta_{\{x_2=0\}}$ in $\mathbb{R}^2$ say, so

$$\langle u, \varphi \rangle = \int \varphi(x_1, 0)\, dx_1$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Clearly $WF(u) = \{(x_1, 0) : (\xi_1, \xi_2)\}$ for some $\{(\xi_1, \xi_2) \neq 0\}$ by considerations of smoothness and one might also guess, since singularities occur as you cross the x_1 axis, that the directions $\{(\xi_1, \xi_2)\}$ in the wave front set are those normal to it. With this in mind, the jump of u across the x_1-axis can be thought of as a *wave front* of the distribution, hence the title.

The above intuition can be substantiated formally, since for any suitable ρ with $\rho(x) = 1$ for some $x = (x_1, 0)$ we have

$$\begin{aligned} \langle f, \rho\, \mathrm{e}^{-\mathrm{i}x.\xi} \rangle &= \int_{\mathbb{R}} \rho(x_1, 0)\, \mathrm{e}^{-\mathrm{i}x_1\xi_1}\, dx_1 \\ &= \widehat{\rho}(\xi_1, 0) \end{aligned}$$

where $\widehat{\rho}(\xi_1, 0)$ is the Fourier transform of $\phi(\cdot, 0)$ in the x_1 variable. But this is less than or equal to $|\xi_1|^{-N} C_N$ for all N if and only if $\xi_1 \neq 0$, so we have

$$WF(u) = \{((x_1, 0), (0, \xi_2)) : \xi_2 \neq 0\}$$

as expected.

We can extend this idea to show that for D a domain in $\mathbb{R}^n$ with smooth boundary ∂D, the indicator function χ_D of D has

$$WF(\chi_D) = \{(x, \xi) : x \in \partial D, \xi \text{ normal to } \partial D\}$$

and more generally, that if $f \in L^1_{\mathrm{loc}}$ is smooth up to a surface $\Sigma = \{x : \phi(x) = 0\,, \phi \in C^\infty(\mathbb{R}^n)\}$ from both sides then

$$WF(f) \subset \left\{(x, \xi)\, :\, x \in \Sigma,\, \xi = t\frac{\partial \phi}{\partial x}(x),\, t \in \mathbb{R} \setminus \{0\}\right\}.$$

To prove this, you simply choose local coordinates which reduce the problem to the case of a function jumping over a hypersurface of the form $\{x \in \mathbb{R}^n\, :\, x_1 = 0\}$ and from here the relevant fourier transforms can be calculated and estimated easily.

However, one must be careful with more complicated distributions. For example, if you consider the wavefront set of χ_S where $S = \{|x_1| \leq 1, |x_2| \leq 1\} \subset \mathbb{R}^2$ is a square (so without a smooth boundary) you discover that it consists of points (x, ξ) with x on the interior of an edge and ξ normal to the edge as usual, but also of the corner points x with corresponding ξ taking any direction!

5.1.4 Effect of Partial Differential Operators

In order to discuss propagation of singularities, in particular to study $WF(u)$ for u a solution to some partial differential equation, we must first consider how applying a linear partial differential operator affects the wave front set of a distribution. If the operator has smooth coefficients we have the following pleasing result:

Theorem 5.9 *If P is a linear partial differential operator with smooth coefficients and $f \in \mathcal{D}'(\mathbb{R}^n)$, then*

$$WF(Pf) \subset WF(f).$$

Proof. We will begin by considering the effect of multiplication by a smooth funtion $\psi \in C^\infty(\mathbb{R}^n)$. Once we have shown that $WF(\psi f) \subset WF(f)$ the extension to arbitrary linear operators with smooth coefficients is relatively straightforward.

Assume $(x_0, \xi_0) \notin WF(f)$, so we would like to show that $(x_0, \xi_0) \notin WF(\psi f)$. By definition, there exists $\rho \in C_0^\infty(\mathbb{R}^n)$ and N a conic neighbourhood of ξ_0 such that $\widehat{\rho f}(\xi)$ satisfies (5.1) for $\xi \in N$. Taking

$\rho' \in C_0^\infty(\mathbb{R}^n)$ with $\rho' = 1$ on $\mathrm{supp}(\rho)$ and letting $\psi' = \rho'\psi$ we have

$$\begin{aligned}
\widehat{\rho(\psi f)}(\xi) &= \widehat{\psi' \rho f}(\xi) && (5.21)\\
&= \widehat{\rho f} \star \widehat{\psi'} && (5.22)\\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\psi'}(\eta)\widehat{\rho f}(\xi - \eta)\, d\eta && (5.23)
\end{aligned}$$

where the first line holds since $\psi = \psi'$ on $\mathrm{supp}(\rho)$ and the second follows from the convolution formula

$$\widehat{f \star g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

If $f \in L^1_{\mathrm{loc}}$ then $|\widehat{\rho f}(\xi)| \le C$, since ρ has compact support implies $\rho f \in L^1$. Indeed, for any distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ the bound

$$|\widehat{\rho f}(\xi)| \le C(1 + |\xi|)^{N_0}$$

holds for some $N_0 \in \mathbb{N}$. This is because, by definition of $\mathcal{D}'(\mathbb{R}^n)$, for any compact set K there exist constants C, N such that

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le N} \sup |\partial^\alpha \varphi|$$

whenever $\varphi \in \mathcal{D}(\mathbb{R}^n)$ has support contained in K. So letting K be a compact set containing $\mathrm{supp}(\rho)$ and choosing appropriate constants C_0, N_0 results in

$$\begin{aligned}
|\widehat{\rho f}(\xi)| &= |\langle f, \rho\, \mathrm{e}^{-\mathrm{i}x.\xi} \rangle| \\
&\le C_0 \sum_{|\alpha| \le N_0} \sup |\partial^\alpha \rho\, \mathrm{e}^{-\mathrm{i}x.\xi}| \\
&\le C(1 + |\xi|)^{N_0}
\end{aligned}$$

for some $C > 0$.

Then, since $\psi' \in C_0^\infty(\mathbb{R}^n)$ and thus satisfies (5.1), we can bound the final expression in (5.21). For any $\delta > 0, R \in \mathbb{N}$ we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \widehat{\psi'}(\eta)\widehat{\rho f}(\xi - \eta)\, d\eta \right| &\le \left| \int_{\delta|\xi| > |\eta|} \widehat{\psi'}(\eta)\widehat{\rho f}(\xi - \eta)\, d\eta \right| + \left| \int_{\delta|\xi| < |\eta|} \widehat{\psi'}(\eta)\widehat{\rho f}(\xi - \eta) \right| \\
&\le \sup_{\delta|\xi| > |\eta|} |\widehat{\rho f}(\xi - \eta)| \int_{\mathbb{R}^n} |\widehat{\psi'}(\eta)|\, d\eta + C \int_{\delta|\xi| < |\eta|} (1 + |\eta|)^{-R} (1 + |\xi - \eta|)^{N_0}\, d\eta
\end{aligned}$$

where in the second integral of the final expression we have used the estimates (5.1) to bound $|\widehat{\psi'}(\eta)|$ and the immediately preceeding remark to bound $|\widehat{\rho f}(\xi - \eta)|$.

For any $M > 0$, one can check that letting $R = M + N_0 + n$ results in an upper bound on the second integral of the form $\tilde{C}_M(1 + |\xi|)^{-M}$ for some constant $\tilde{C}_M$.

To bound the first integral, observe that since N is a conic neighbourhood of ξ_0, there exists a $\beta > 0$ such that

$$\left| \frac{\xi - \eta}{|\xi - \eta|} - \frac{\xi_0}{|\xi_0|} \right| < \beta \Rightarrow \xi - \eta \in N.$$

To see this, observe that N must contain ξ_0 and therefore $\frac{\xi_0}{|\xi_0|}$ (as it is conic), and since it is a neighbourhood there must exist some $\beta > 0$ such that $\left| (\xi - \eta) - \frac{\xi_0}{|\xi_0|} \right| < \beta \Rightarrow \xi - \eta \in N$. From here the result follows by using the conic property once more.

Given such a β, if $|\eta|$ is small enough compared to $|\xi|$ we will have that

$$\left|\frac{\xi-\eta}{|\xi-\eta|}-\frac{\xi}{|\xi|}\right|<\frac{\beta}{2} \tag{5.24}$$

and so we can choose $\delta<\frac{1}{2}$ such that $\delta|\xi|>|\eta|$ implies that (5.24) holds.

Then if

$$\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\frac{\beta}{2},$$

by the triangle inequality we have $\delta|\xi|>|\eta| \Rightarrow \left|\frac{\xi-\eta}{|\xi-\eta|}-\frac{\xi_0}{|\xi_0|}\right|<\beta$ and so

$$\sup_{\delta|\xi|>|\eta|}\left|\widehat{\rho f}(\xi-\eta)\right| \leq C_M\left(1+|\xi|\right)^{-M}$$

as required.

Therefore, $\left\{\xi\,:\,\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\frac{\beta}{2}\right\}$ defines a conic neighbourhood on which, for all M, we can bound both the terms in our expression for $|\widehat{\rho(\psi f)}(\xi)|$ by $C_M(1+|\xi|)^{-M}$. Thus, $|\widehat{\rho(\psi f)}(\xi)|$ obeys the estimates (5.1) on this neighbourhood, and so $(x_0,\xi_0)\notin WF(\psi f)$ and $WF(\psi f)\subset WF(f)$ as desired.

From here, the extension to an arbitrary linear partial differential operator with smooth coefficients

$$Pf=\sum_{|\alpha|\leq m}\psi_\alpha(x)\,\partial^\alpha f$$

is relatively straightforward. We have the identity

$$\widehat{\frac{\partial\rho f}{\partial x_j}}(\xi)=\mathrm{i}\xi_j\widehat{\rho f}(\xi)$$

and so if $\widehat{\rho f}(\xi)\leq C_N(1+|\xi|)^{-N}$ then

$$\begin{aligned}\left|\widehat{\frac{\partial\rho f}{\partial x_j}}(\xi)\right| &= \left|\xi_j\widehat{\rho f}(\xi)\right| \\ &\leq C_N|\xi_j|\left(1+|\xi|\right)^{-N} \\ &\leq C_N\left(1+|\xi|\right)^{-N+1}\end{aligned}$$

where the last line holds since $|\xi_j|\leq|\xi|\leq 1+|\xi|$ for all ξ. This means that for each α, $\left|\widehat{\rho\partial^\alpha f}(\xi)\right|$ satisfies estimates of the form (5.1) and so also, by previous work, does $\left|\widehat{\rho(\psi_\alpha\,\partial^\alpha f)}(\xi)\right|$. Since Pf is just the sum of terms $\psi_\alpha\,\partial^\alpha f$ and the Fourier transform is linear, the result follows immediately. $\square$

Elliptic regularity results give us that when P is an elliptic operator, the inclusion in the above theorem is actually an equality. The key result from propagation of singularities is that when P is strictly hyperbolic, it may well be a strict inclusion. In the next section, as discussed in the introduction, we will show that in this case singularities must actually propagate along specific curves (null bicharacteristics) corresponding to the operator.

5.2 Propagation of Singularities

We will now use the gaussian beam construction to prove a propagation of singularities result for strictly hyperbolic differential operators. The object of our attention will be the behaviour of singularities of u, the solution to the mixed problem:

$$\begin{aligned} Pu &= f && \text{on } \mathbb{R}_+ \times \Omega \\ B_i u &= g_i && \text{on } \mathbb{R}_+ \times\ \partial\Omega \ \ i = 1, \cdots, l \\ \frac{\partial^i u}{\partial t^i} &= h_i && \text{on } \{0\} \times \Omega \ \ i = 0, \cdots, m-1 \end{aligned} \tag{5.25}$$

where the B_i's are some linear differential operators. From now on we will denote elements of $\mathbb{R}_+ \times \Omega$ by $(t, x_1, \cdots, x_n)$, and directly apply the results of the previous section with this change of notation.

In fact, we will actually use the gaussian beam construction for the *adjoint* operator P^*. By this we mean by this the unique differential operator such that

$$\int_{\mathbb{R}^{n+1}} \overline{v} Pu = \int_{\mathbb{R}^{n+1}} \overline{P^*} vu$$

for all $u, v \in C_0^\infty(\mathbb{R}^{n+1})$. Observe that since they are real valued, the principle symbols of P and P^* coincide. In particular, this means that P^* is strictly hyperbolic, and so subject to the necessary assumptions, our gaussian beam construction is admissable for P^*. To formulate the corresponding mixed initial-boundary value problem for P^* we also need some adjoint boundary conditions

$$B_i^* v = 0 \ \ i = l+1, \cdots m$$

to those given in (5.25). These are defined to be linear operators such that

$$\int_{\mathbb{R}\times\Omega} \left(\overline{v}Pu - \overline{P^* v}u\right)\, dx = \int_{\mathbb{R}\times\ \partial\Omega} \left(\sum_{i=1}^{l} \overline{C_i v} B_i u + \sum_{i=l+1}^{m} \overline{B_i^* v} C_i u \right) dx$$

holds for some linear operators $C_i, i = 1, \cdots, m$ and all $u, v \in C_0^\infty(\mathbb{R}^{n+1})$. The simplest example, and we will use it later on, is when $B_i u = \frac{\partial^{i-1} u}{\partial \nu^{i-1}}$ for $i = 1, \cdots l$ and $\frac{\partial}{\partial \nu}$ is the normal derivative to the boundary $\partial\Omega$. In this case, we can obtain by integration by parts that the adjoint operators are given by $B_i^* v = \frac{\partial^{i-1} v}{\partial \nu^{i-1}}$, $i = 1, \cdots, m-l$.

As is a standard approach in PDEs when working with the adjoint operator, we will consider solutions starting at some fixed time $T > 0$ and extending into $t < T$. In our case, this means considering ray paths starting at a point $(\underline{x}, T)$ and moving backwards with respect to time.

So, given $(\underline{x}, \underline{\xi}) \in \text{Int}(\Omega) \times S^{n-1}$, and noting the new representation $p_m(t, x, \tau, \xi)$ for the principle symbol corresponding to the distinction of the time variable, we let $\{\tau_i, i = 1, \cdots m\}$ be the roots of $p_m(T, \underline{x}, \tau, \underline{\xi}) = 0$ and follow the null bicharacteristics eminating from $(T, \underline{x}, \tau_i, \underline{\xi})$ backwards into $t < T$. Whenever we reach a point (t', x', τ', ξ') with x' on the boundary $\partial\Omega$, as in the construction of reflected gaussian beams, we then move along null bicharacteristics starting from (t', x', τ', ξ'_i) where $\xi_i = \xi' + s_i \nu$ are the real roots of $p_m(t', x', \tau', \xi' + s\nu) = 0$ such that

$$0 > \nu \cdot \frac{\partial p_m}{\partial \xi} \frac{\partial p_m}{\partial \tau}\Big|_{(t', x', \tau', \xi' + s_i \nu)} \tag{5.26}$$

and ν is the inner unit normal to $\partial\Omega$. Observe the sign in the condition (5.26) is opposite to that in the reflected gaussian beam construction. In that case, the condition ensured that the ray paths would propagate forwards in time, and applying the same reasoning we get the opposite from (5.26), which is precisely what we want. If we reach a point where there are no admissable s_i then we stop, otherwise continue until reaching time $t = 0$, and end up with a collection of points

$$(0, y_i, \tau_i, \xi_i)$$

at the end of the ray paths. These are the time 0 data for the *shower* produced by tracing backwards from $(T, \underline{x}, \underline{\xi})$.

We make the following assumptions on this procedure:

- all bicharacteristics in the shower never graze. So for all (t', x', τ', ξ') in the shower with $x' \in \partial\Omega$, the polynomial $p_m(t', x', \tau', \xi' + s\nu) = 0$ has m distinct roots in $\mathbb{C}$.
- all bicharacteristics in the shower either stop (in the case that there are no admissable s_i's) or reach $t = 0$ in a finite number of reflections.
- $y_i \in \text{Int}(\Omega)$ for $i = 1, \cdots M$.
- the adjoint boundary conditions satisfy the second assumption we required for the construction of reflected gaussian beams, ie. the matrix $(b^*)_{ij}$ corresponding to the B_i^*'s has rank $m - l$.

Observe that these assumptions will allow us to construct reflected Gaussian beams for the adjoint problem, which will be our key tool.

Given the above, the following theorem holds:

Theorem 5.10 (Propagation of Singularities) *Suppose $u \in H^m([0, T] \times \Omega)$ satisfies (5.25) and*

- $(y_i, \xi_i) \notin WF(h_j)$ *for* $i = 1, \cdots, M$ *and* $j = 0, \cdots m - 1$.
- *For all (t', x', τ', ξ') in the shower tracing back from $(T, \underline{x}, \underline{\xi})$ with $x' \in \partial\Omega$, the g_j are smooth in a neighbourhood of (t', x') for $j = 1, \cdots, l$.*
- *supp*$(f) \subset$ *Int*$([0, T] \times \Omega)$ *and $WF(f)$ does not intersect the shower.*

Then $(\underline{x}, \underline{\xi}) \notin WF\left(\frac{\partial^r u}{\partial t^r}(T, \cdot)\right)$ *for* $r = 0, 1, \cdots, m - 1$ *and hence from (5.25), for all* r.

Recalling Theorem 5.7, this means that $\underline{x}$ can only be a singularity of u at time T if, tracing back along the null bicharacteristics from $(T, \underline{x}, \underline{\xi})$ for some direction $\underline{\xi}$, there is a singularity of f at an earlier time or of the initial/boundary data. So, as alluded to earlier, singularities can only propagate along ray paths. Note also that we have used a smoothness hypothesis for the g_j's in the statement of the theorem, rather than an assumption on their wave front sets. The theorem extends to the latter case, as has been shown by Hörmander [1], but requires definition of the wave front set for distributions on a manifold (ie. $\mathbb{R} \times \partial\Omega$.) In this case, the wave front set can be defined as a subset of the cotangent bundle, but we will not go into details here.

Proof. To begin, choose $\delta > 0$ small enough so that whenever $|\xi - \underline{\xi}| < \delta$ and $|x' - \underline{x}| < \delta$, the properties of the shower obtained tracing back from (T, x', ξ) are the same as those of the shower from $(T, \underline{x}, \underline{\xi})$. Then, for any (x', ξ) each within δ of $(\underline{x}, \underline{\xi})$ apply the results of the previous section for the adjoint problem. That is, given N, construct Gaussian beams $\omega(t, x; x', \xi, k)$ (corresponding to null bicharacteristics starting at (T, x', τ_j, ξ) for $\{\tau_j, j = 1, \cdots m\}$ the roots of $p_m(T, x', \tau, \xi)$ and evolving backwards in time) such that

(i) $\|P^*\omega\|_0 \leq Ck^{-N}$

(ii) $\|B_i^*\omega\|_0 \leq Ck^{-N}$ $\;i = l + 1, \cdots, m$

(iii) $\left\|\frac{\partial^r \omega}{\partial t^r}(T, x)\right\|_0 \leq Ck^{-N}$ $\;r = 0, \cdots, m - 2$ and
$\left\|\frac{\partial^{m-1}\omega}{\partial t^{m-1}}(T, x, x', \xi) - k^{m-1}\phi(x)\exp\left(\mathrm{i}kx.\xi - \frac{k}{2}|x - x'|^2\right)\right\|_0 \leq Ck^{-N}$

for $\phi \in C_0^\infty(|x - \underline{x}| < \delta)$ with $\phi(\underline{x}) = 1$.

So for time close to T, before any reflections have occured, ω is of the form

$$\omega = \sum_{j=1}^{m} \mathrm{e}^{\mathrm{i}k\psi_j} \left(a_0^j + \cdots + \frac{a_R^j}{k^R} \right) \tag{5.27}$$

for some r, where

$$\left(\frac{\partial \psi_j}{\partial t}(T, x'), \frac{\partial \psi_j}{\partial x}(T, x') \right) = (\tau_j(x', \xi), \xi)$$

and the τ_j's are as described above.

The assumptions we've made ensure that the gaussian beam construction as in the previous section is admissable for P^*, and so we need only check that it is possible to obtain condition (iii). Recall from the construction, that to ensure closeness of ω to the given initial data (in this case meaning at time $t = T$), we need only prescribe the Taylor series of the coefficients $a_r^j(T, x)$ at fixed time T, about the point x' appropriately. Differentiating i times ($i = 0, \cdots, m-1$) the expression (5.27) with respect to time, evaluating at T and either insisting that this vanishes to high order at x' or agrees with $k^{m-1}\phi(x) \exp\left(\mathrm{i}kx.\xi - \frac{k}{2}|x - x'|^2\right)$ for $i = m-1$, we see that the Taylor series is determined by equations of the form

$$\sum_{j=1}^{m} (\tau_j(x', \xi))^i \, a_r^j(T, x') = g_{ri} \tag{5.28}$$

for each i, r. Here the g_{ri}'s are determined by ϕ, the ψ_j, and $a_{r'}^j$ for $r' < r$.

So we see that we must solve inductively in r systems of the form

$$A a_r = g_r \tag{5.29}$$

where $A_{ij} = (\tau_j(x, \xi))^i$, $(a_r)_j = a_r^j(T, x)$ and $(g_r)_i = g_{ri}$ as above.

However, by the strict hyperbolicity assumption, the τ_j's are distinct and so the matrix A is a Vandermonde matrix. This means that the above system (5.29) is uniquely solvable, and so our construction of ω is justified. Referring back to the previous section we also know that ω is smoothly dependent on (x', ξ), and so provided δ is small enough we may assume that the constant C in (i)-(iii) is uniform in $|\underline{x} - x'| < \delta$, $|\underline{\xi} - \xi| < \delta$.

Now, since $\underline{x}$ and the y_i's lie in the interior of Ω by assumption, and ω is concentrated around the (x, t) projection of the shower tracing back from (T, x, ξ) to the (y_i, ξ_i)'s, we can take ω to vanish near the corners $\{T\} \times \partial\Omega$ and $\{0\} \times \partial\Omega$. Applying integration by parts, since $u \in H^m([0, T] \times \Omega)$, we have

$$\begin{aligned}
\int_{[0,T]\times\Omega} \overline{\omega} f &= \int_{[0,T]\times\Omega} \overline{\omega} P u \\
&= \int_{[0,T]\times\Omega} \overline{P^*\omega} u + \int_{[0,T]\times\,\partial\Omega} M(u, \omega, \nu) \\
&+ \int_{\{0\}\times\Omega} \tilde{M}(u, \omega, \mathbf{e}_t) + \int_{\{T\}\times\Omega} \tilde{\tilde{M}}(u, \omega, -\mathbf{e}_t)
\end{aligned} \tag{5.30}$$

where ν is the normal to $\partial\Omega$ and $M, \tilde{M}, \tilde{\tilde{M}}$ are expressions determined by the order in which we do the integration by parts. Observe the absence of boundary terms from the corners, due to the assumption on ω made immediately before.

By definition of the adjoint operators B_i^* we see that

$$\int_{[0,T]\times\,\partial\Omega} M(u, \omega, \nu) = \int_{[0,T]\times\,\partial\Omega} \left(\sum_{i=1}^{l} \overline{C_i \omega} B_i u + \sum_{i=l+1}^{m} B_i^* \omega C_i u \right) \tag{5.31}$$

where on $\partial\Omega$ we have $B_i u = g_i$ for each i by assumption. Furthermore, recall from the previous discussion that the adjoints of operators $B_i u = \frac{\partial^{i-1} u}{\partial \nu^{i-1}}$, $i = 1, \cdots, l$ are given by $B_i^* u = \frac{\partial^{i-1} u}{\partial \nu^{i-1}}$, $i = l+1, \cdots, m$. Applying this with $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial t}$ in the case $l = m$ gives

$$\int_{\{0\}\times\Omega} \tilde{M}(u, \omega, \mathbf{e}_t) = \int_{\{0\}\times\Omega} \sum_{i=1}^{m} \overline{E_i \omega} h_i$$

($h_i = \frac{\partial^{i-1} u}{\partial t^{i-1}}$ on this domain) and with $l = 0$ gives

$$\int_{\{T\}\times\Omega} \tilde{M}(u, \omega, -\mathbf{e}_t) = \int_{\{T\}\times\Omega} \sum_{i=1}^{m} \overline{\frac{\partial^{i-1}\omega}{\partial t^{i-1}}} D_i u$$

for some differential operators E and D.

Although we will not need much information about these operators we can calculate, simply proceeding with the integration by parts, that since the coefficient of $\frac{\partial^m}{\partial t^m}$ in P is assumed to be one, we have

$$D_m u = (-1)^{m-1} u.$$

Putting the above together and rewriting (5.31) with the equivalent expressions for each term, we obtain

$$\begin{aligned}
\int_{[0,T]\times\Omega} \overline{\omega} f \quad &= \quad \int_{[0,T]\times\Omega} \overline{P^*\omega} u && (5.32)\\
&+ \quad \int_{[0,T]\times\,\partial\Omega} \left(\sum_{i=1}^{l} \overline{C_i \omega} g_i + \sum_{i=l+1}^{m} \overline{B_i^* \omega} C_i u \right) && (5.33)\\
&+ \quad \int_{\{0\}\times\Omega} \sum_{i=1}^{m} \overline{E_i \omega} h_i && (5.34)\\
&+ \quad \int_{\{T\}\times\Omega} \sum_{i=1}^{m} \overline{\frac{\partial^{i-1}\omega}{\partial t^{i-1}}} D_i u && (5.35)
\end{aligned}$$

By condition (i) in the construction of ω, we have that (5.32) is $O(k^{-N})$, and by condition (iii) along with the observation that $D_m u = (-1)^{m-1} u$, (5.35) is equal to

$$(-1)^{m-1} \int_{\{T\}\times\Omega} k^{m-1} \phi(x)\, \mathrm{e}^{\mathrm{i}kx.\xi - \frac{k}{2}|x-x'|^2}\, u(x,T)\, dx + O(k^{-N}).$$

Furthermore, since our constants in (i)-(iii) were uniform in $|x' - \underline{x}| < \delta$ and $|\xi - \underline{\xi}| < \delta$, these $O(k^{-N})$ are uniform for x' in a neighbourhood of supp(ϕ) ($\phi \in C_0^\infty(|x - \bar{x}| < \delta)$) and $|\xi - \xi'| < \delta$. To estimate (5.33) and (5.34) similarly, we need the following Lemma:

Lemma 5.11 *Assume* $\frac{\partial\psi}{\partial x} \neq 0$ *and* $\left(\frac{\partial\psi}{\partial x}\right)(x_0) = \xi_0$ *with* $(x_0, -\xi_0) \notin WF(u)$ *for* $u \in L^2_{loc}$. *Assume* $\phi \in C_0^\infty(\mathbb{R}^n)$ *and* $\mathrm{Im}\psi \geq c^2|x - x_0|^2$ *on the support of* ϕ. *Then there are constants* C_N *such that*

$$\left| \int \mathrm{e}^{\mathrm{i}k\psi}\, \phi u\, dx \right| \leq C_N k^{-N} \qquad (5.36)$$

for $k > 0$ *and* $N \in \mathbb{Z}$.

Moreover, if ψ *depends smoothly on parameters* y, η, *so*

$$\psi = \psi(x; y, \eta) \text{ and } \frac{\partial\psi}{\partial x}(x_0; y_0, \eta_0) = \xi_0,$$

then the constants in (5.36) are uniform for (y, η) *in* $|y - y_0| < \delta_0$ *and* $|\eta - \eta_0| < \delta_0$.

We will use this for now without a proof, which will be postponed to the end of the section.

The Lemma can be applied immediately to show that (5.34) is $O(k^{-N})$ (uniformly for x' in a neighbourhood of $\mathrm{supp}(\phi)$ and $|\xi - \underline{\xi}| < \delta$). This is because, for each i in the sum, $\overline{E_i\omega}h_i$ is an expression of the form

$$\sum_j \phi_j \, \mathrm{e}^{-\mathrm{i}k\bar{\psi}_j}\, h_i \tag{5.37}$$

where $\phi_j \in C_0^\infty(\mathbb{R}^n)$ and each ψ_j comes from the expression for the reflected gaussian beams (the sum is over all ray paths in the shower tracing back from (T, x', ξ)), evaluated at time $t = 0$. Observe that we have $\mathrm{e}^{-\mathrm{i}k\bar{\psi}_j}$ here, rather than $\mathrm{e}^{\mathrm{i}k\psi_j}$ as in the expression for ω, which is due to the fact that we have complex conjugated ω in (5.34). Since these ray paths end at the points (y_j, ξ_j) for each j, by construction of the gaussian beams we have $\mathrm{Im}(-\bar{\psi}_j) = \mathrm{Im}(\psi_j) \geq c^2|x - y_j|^2$ on $\mathrm{supp}(\phi_j)$ and $\left(\frac{\partial(-\bar{\psi}_j)}{\partial x}\right)(y_j) = -\xi_j$. Moreover, by the non-grazing hypothesis $\frac{\partial(-\bar{\psi}_j)}{\partial x} \neq 0$ and finally by our original assumptions, $(y_j, -(-\xi_j)) = (y_j, \xi_j) \notin WF(h_i)$ for each i, j. Thus, applying the Lemma directly gives us the required result.

A little more care is required for the term (5.33), although the principle is the same. The term $\int_{[0,T]\times\,\partial\Omega} \sum_{i=l+1}^{m} B_i^*\omega C_i u$ is $O(k^{-N})$ by condition (ii) for ω, and so we need only deal with

$$\int_{[0,T]\times\,\partial\Omega} \sum_{i=1}^{l} \overline{C_i\omega} g_i.$$

The slight difficulty here arises from the fact that we are working on the boundary $\partial\Omega \times [0, T]$. To deal with this we introduce local coordinates $y_1, \cdots, y_{n-1}$ on $\partial\Omega$ and write $g_i = g_i(t, x(y))$. As above, we can then write the expression inside the integral as

$$\sum_j \phi_j \, \mathrm{e}^{\mathrm{i}k\psi_j}\, g_i$$

for phase functions $\psi_j = \psi_j(t, x(y))$, where everything in the Lemma holds trivially by construction of the gaussian beams except that $\left(\frac{\partial\psi_j}{\partial t}, \frac{\partial\psi_j}{\partial y}\right) \neq 0$. The wave front set condition comes from the assumption that the g_j's are smooth at all points (t', x') such that (t', x', τ', ξ') are on bicharacterstics in the shower with $x' \in \partial\Omega$. Clearly this implies that $(t', x'(y), -\frac{\partial\psi_j}{\partial t}, -\frac{\partial\psi_j}{\partial y}) \notin WF(g_i)\,\forall i$. To verify that $\left(\frac{\partial\psi_j}{\partial t}, \frac{\partial\psi_j}{\partial y}\right) \neq 0$ note that $\left(\frac{\partial x}{\partial y}\right)$ has rank $n-1$ and $\nu \cdot \frac{\partial x}{\partial y} = 0$ since the y_k's are coordinates on $\partial\Omega$ and $\nu \perp \Omega$. Therefore

$$\left(\frac{\partial\psi_j}{\partial t}, \frac{\partial\psi_j}{\partial y}\right) = 0 \Leftrightarrow \left(\frac{\partial\psi_j}{\partial t}\frac{\partial\psi_j}{\partial y}\right) = (0, \alpha\nu)$$

for some α, but this is not allowed as a result of the non grazing hypothesis. Hence, we may apply the lemma to deduce that the (5.36) is $O(k^{-N})$ uniformly for x' in a neighbourhood of $\mathrm{supp}(\phi)$ and $|\xi - \underline{\xi}| < \delta$.

Putting this all together we obtain that

$$\int_{[0,T]\times\Omega} \bar{\omega} f = (-1)^{m-1} \int_\Omega k^{m-1}\phi(x) \exp\left(-\mathrm{i}kx.\xi - \frac{k}{2}|x - x'|^2\right) \times u(x, T)\, dx + O(k^{-N}) \tag{5.38}$$

where $O(k^{-N})$ is uniform on the set described above.

Moreover, by modifying the Lemma slightly (as will be discussed in the proof) since it was assumed that the wave front set of f does not intersect the shower, we get that for given $t_0 \in (0, T)$ there exists an $\varepsilon > 0$ such that for $\rho \in C_0^\infty(|t - t_0| < \varepsilon)$

$$\left|\int_{[0,T]\times\Omega} \rho\bar{\omega} f\right| \leq C_N k^{-N}. \tag{5.39}$$

Using the two expressions (5.38) and (5.39) above, we see that

$$\begin{aligned}\int_\Omega \phi(x)\exp\left(-\mathrm{i}kx\cdot\xi-\frac{k}{2}|x-x'|^2\right)u(x,T)\,dx &= (-1)^{m-1}\int_{[0,T]\times\Omega}\bar{\omega}f+O(k^{-N})\\ &\le C_N k^{-N} \qquad (5.40)\end{aligned}$$

where the final inequality holds since we claim that we can find constants $\tilde{C}_N$ such that $\int_{[0,T]\times\Omega}\bar{\omega}f \le \tilde{C_N}k^{-N}$.

One can verify the claim as follows: take ε_t as in (5.39) for each $t\in(0,T)$ and form an open cover $\{(t-\varepsilon_t,t+\varepsilon_t): t\in[\delta',T-\delta']\}$ of $[\delta',T-\delta']$ where $\delta'>0$ is chosen so that $\mathrm{supp}(f)\subset[\delta',T-\delta']\times\Omega$. This choice of δ' is permissable due to the assumption that the support of f lies in the interior of $[0,T]\times\Omega$. Let $\{(t_0^i-\varepsilon_{t_0^i},t_0^i+\varepsilon_{t_0^i}) : i\in I\}$ be a finite subcover and then take a partition of unity $\{\rho_i\}_{i\in I}$ such that each ρ_i belongs to $C_0^\infty(|t-t_0^i|<\varepsilon_{t_0^i})$ for some i. Suppose in (5.39) we have constants C_N^i corresponding to t_0^i for each $i\in I$. Then

$$\begin{aligned}\left|\int_{[0,T]\times\Omega}\bar{\omega}f\right| &= \left|\int_{[0,T]\times\Omega}\sum_I\rho_i\bar{\omega}f\right|\\ &\le \sum_I\left|\int_{[0,T]\times\Omega}\rho_i\bar{\omega}f\right|\\ &\le \sum_I C_N^i k^{-N}\\ &= \tilde{C}_N k^{-N}\end{aligned}$$

for some constants $\tilde{C}_N$, where the first equality holds since $\{\rho_i\}_{i\in I}$ is a partition of unity of $[\delta',T-\delta']$ and f has support contained in the interior of $[\delta',T-\delta']\times\Omega$ by assumption.

As usual, we can choose the constants C_N in (5.40) to be uniform in a neighbourhood of $\mathrm{supp}(\phi)$, which we'll denote by $\mathcal{O}$, and for $|\xi-\underline{\xi}|<\delta$. Thus, multiplying by $k^{n/2}$ and integrating over all $x'\in\mathcal{O}$ we see that

$$\left|\int_\Omega\phi(x)\,\mathrm{e}^{-\mathrm{i}kx.\xi}\left(k^{n/2}\int_{\mathcal{O}}\mathrm{e}^{-k/2|x-x'|^2}\,dx'\right)u(x,T)\,dx\right|\le C_N k^{-N+\frac{n}{2}}.$$

Note that from now on the constants C_N will vary from line to line.

If we let $C=\int_{\mathbb{R}^n}\mathrm{e}^{-\frac{|y|^2}{2}}\,dy$ and make the change of variables $y=\sqrt{k}(x'-x)$, we obtain that

$$\begin{aligned}\left|\phi(x)\left(C-k^{n/2}\int_{\mathcal{O}}\mathrm{e}^{-k/2|x-x'|^2}\,dx'\right)\right| &= \left|\phi(x)\left(k^{n/2}\int_{\mathbb{R}^n\setminus\mathcal{O}}\mathrm{e}^{-k/2|x-x'|^2}\,dx'\right)\right|\\ &\le C_N k^{-N}\end{aligned}$$

uniformly for $x\in\mathbb{R}^n$. This holds since the expression is only non-zero for $x\in\mathrm{supp}(\phi)$ and the region of integration (outside of $\mathcal{O}$ a neighbourhood of $\mathrm{supp}(\phi)$) is a fixed distance away from x. But from here one can easily deduce, using the triangle inequality, that

$$\begin{aligned}C\left|\int_\Omega\phi(x)\,\mathrm{e}^{-\mathrm{i}kx.\xi}\,u(x,T)\,dx\right| &\le \left|\int_\Omega\phi(x)\exp\left(-\mathrm{i}kx\cdot\xi-\frac{k}{2}|x-x'|^2\right)u(x,T)\,dx\right|\\ &\quad+\left|\int_\Omega\phi(x)\left(C-k^{n/2}\int_{\mathcal{O}}\mathrm{e}^{-k/2|x-x'|^2}\,dx'\right)\mathrm{e}^{-\mathrm{i}kx.\xi}\,u(x,T)\,dx\right|\\ &\le C_N k^{-N+n/2}\end{aligned}$$

where the bound on the first term comes from (5.40) and the second from the above and the fact that $e^{-ikx.\xi}$ belongs to L^1 (plus multiplying by $k^{n/2}$ which is allowed since it increases the bound.)

Thus, altering and relabelling the constants $C_N \leftrightarrow C_{N-n/2}$, we reach the conclusion that

$$\left| \int_\Omega \phi(x)\, e^{-ikx.\xi}\, u(x,T)\, dx \right| \le C_N k^{-N} \tag{5.41}$$

for each N, uniformly for $|\xi - \underline{\xi}| < \delta$. Hence, letting $k = |\xi|$ and applying the above to $\frac{\xi}{|\xi|}$, we get that

$$\left| \widehat{\phi u(T,\cdot)}(\xi) \right| = \left| \int_\Omega \phi(x)\, e^{-ix.\xi}\, u(x,T)\, dx \right| \le C_N |\xi|^{-N}$$

for all $\xi \in N = \left\{ \xi \,:\, \left| \frac{\xi}{|\xi|} - \underline{\xi} \right| \right\}$, a conic neighbourhood of $\underline{\xi}$. Recalling the definition of the wavefront set, along with the subsequent remarks, this implies that $(\underline{x}, \underline{\xi}) \notin WF(u(x,T))$.

So now, all that is left is to show that $(\underline{x}, \underline{\xi}) \notin WF\left(\frac{\partial^r u}{\partial t^r}(T,\cdot)\right)$ for $r = 0, \cdots, m-1$. We'll first consider the case $r = 1$, and return to the expression

$$\begin{aligned} \int_{[0,T]\times\Omega} \overline{\omega} f &= \int_{[0,T]\times\Omega} \overline{P^*\omega} u + \int_{[0,T]\times\,\partial\Omega} \left(\sum_{i=1}^{l} \overline{C_i \omega} g_i + \sum_{i=l+1}^{m} B_i^* \omega C_i u \right) \\ &+ \int_{\{0\}\times\Omega} \sum_{i=1}^{m} \overline{E_i \omega} h_i + \int_{\{T\}\times\Omega} \sum_{i=1}^{m} \overline{\frac{\partial^{i-1}\omega}{\partial t^{i-1}}} D_i u. \end{aligned}$$

Considering the integration by parts procedure by which the operators D_i are obtained, we have that

$$D_{m-1} u = (-1)^{m-2} \frac{\partial u}{\partial t} + F(u)$$

where F is some operator which depends only on the x-variables. By the above proof and theorem (5.9),

$$(\underline{x}, \underline{\xi}) \notin WF(u(T,\cdot)) \Rightarrow (\underline{x}, \underline{\xi}) \notin WF(F(u(T,\cdot)))$$

and so we can choose ϕ with $\phi(\underline{x}) = 1$ such that

$$\left| \phi\, e^{-ikx.\xi}\, F(u)(x,T)\, dx \right| \le C_N k^{-N} \tag{5.42}$$

uniformly for $|\xi - \underline{\xi}| < \delta$. Constructing a new Gaussian beam ω such that

$$\begin{aligned} \left\| \frac{\partial^r \omega}{\partial t^r}(T,x) \right\|_0 &\le C_N k^{-N} \;\; 0 \le r \le m-1\,,\, r \ne m-2 \\ \left\| \frac{\partial^{m-2}\omega}{\partial t^{m-2}}(T,x) - k^{m-2}\phi(x) \exp\left(ikx\cdot\xi - k/2|x-x'|^2 \right) \right\|_0 &\le C_N k^{-N} \end{aligned}$$

we can use exactly the previous argument, with $m-1$ in place of m, to show that

$$\left| \int_\Omega \phi(x)\, e^{-ikx.\xi}\, D_{m-1} u(x,T) \right| \le C_N k^{-N}.$$

Substituting in the expression for $D_{m-1}u$ and using the triangle inequality with (5.42) shows that $(\underline{x}, \underline{\xi}) \notin WF(\frac{\partial u}{\partial t}(T,\cdot))$ as required. For $r = 2, \cdots m-1$ we may apply the same reasoning inductively, considering the operator D_{m-r} in each case, to reach the conclusion.

□

In order to complete the proof of the theorem, we must now prove Lemma (5.11).

Recall that we are trying to prove the existence of constants C_N such that

$$\left|\int e^{ik\psi}\,\phi u\,dx\right| \leq C_N k^{-N} \tag{5.43}$$

assuming that $\frac{\partial\psi}{\partial x} \neq 0$, $\left(\frac{\partial\psi}{\partial x}\right)(x_0) = \xi_0$ and $(x_0, -\xi_0) \notin WF(u)$ for $u \in L^2_{\text{loc}}$, along with $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\text{Im}\psi \geq c^2|x-x_0|^2$ on $\text{supp}(\phi)$.

Proof. Firstly, by the assumption that $(x_0, -\xi_0) \notin WF(u)$ there must exist $\rho \in C_0^\infty(\mathbb{R}^n)$ with $\rho \equiv 1$ on a neighbourhood of x_0 such that

$$|\widehat{\rho u}(\xi)| \leq C_N\,(1+|\xi|)^{-N} \tag{5.44}$$

for all ξ satisfying

$$\left|\frac{\xi}{|\xi|} + \frac{\xi_0}{|\xi_0|}\right| < \delta.$$

Here we recall from the previous proof that any conic neighbourhood of $-\xi_0$ contains such a set of ξ's.

Since $\frac{\partial\psi}{\partial x}(x_0) = \xi_0$, by continuity we must have that $\frac{\partial\psi}{\partial x}(x)$ is close to ξ_0 for $|x-x_0|$ small. Furthermore, $|k\xi_0+\xi|^2 > \tilde{c}(k^2+|\xi|^2)$ for $\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| > \frac{\delta}{2}$ and so if ξ is in this set, and $|x-x_0| < \delta'$ for δ' small enough then

$$\left|k\frac{\partial\psi}{\partial x}(x) + \xi\right|^2 > c(k^2+|\xi|^2),\ \ c > 0. \tag{5.45}$$

Define the operator

$$(Bw)(x) = (2\pi)^{-n/2}\int e^{ix.\xi}\,b(\xi)\widehat{w}(\xi)\,d\xi \tag{5.46}$$

where $b(\xi)$ is chosen to vanish when $\left|\frac{\xi}{|\xi|} + \frac{\xi_0}{|\xi_0|}\right| \leq \frac{\delta}{2}$ and when $|\xi| < 1/2$, and be a homogeneous function of degree 0 for $|\xi| > 1$, equal to one when $\left|\frac{\xi}{|\xi|} + \frac{\xi_0}{|\xi_0|}\right| \geq \delta$. This construction is to ensure that $(1-b(\xi))$ vanishes for all ξ such that the estimates (5.44) may fail to hold, and so that (5.45) holds on $\text{supp}(b)$ for all appropriate x.

Choosing $\rho_0 \in C_0^\infty(|x-x_0| < \delta')$ with $\rho_0 \equiv 1$ on a neighbourhood of x_0, write

$$\begin{aligned}\int e^{ik\psi}\,\phi u &= \int e^{ik\psi}(1-\rho_0\rho)\phi u\,dx &(5.47)\\ &+ \int e^{ik\psi}\,\phi\rho_0(\rho u - B(\rho u)) &(5.48)\\ &+ \int e^{ik\psi}\,\phi\rho_0 B(\rho u). &(5.49)\end{aligned}$$

The goal will be to bound each of the above terms by $\{C_N k^{-N} : N \in \mathbb{N}\}$, and then the result will follow.

The first is straightforward since $\rho_0\rho \equiv 1$ on a neighbourhood of x_0 implies that $(1-\rho_0\rho) \equiv 0$ on that neighbourhood and therefore $\text{Im}\psi > r > 0$ on $\text{supp}((1-\rho_0\rho)\phi u)$. Thus the term $e^{ik\psi}$, which has modulus $e^{-k\text{Im}\psi} < e^{-kr}$, allows us to deduce bounds of the required form.

For the second, note that since $(1-b(\xi)) = 0$ for all ξ such that the bounds (5.44) may fail to hold

$$(2\pi)^{-n/2}\int e^{ix.\xi}(1-b(\xi))\widehat{\rho u}(\xi)\,d\xi$$

converges and is equal to $f = \rho u - B(\rho u)$. Using the bounds and differentiating under the integral, we therefore have that $f \in C^\infty(\mathbb{R}^n)$.

Defining an operator L by

$$Lw = \left|\frac{\partial\psi}{\partial x}\right|^{-2} \frac{\partial\psi}{\partial x} \cdot \frac{\partial w}{\partial x},$$

it is trivial to check that $L\,\mathrm{e}^{\mathrm{i}k\psi} = \mathrm{i}k\,\mathrm{e}^{\mathrm{i}k\psi}$, and so for any N (5.48) is equal to

$$\frac{1}{(\mathrm{i}k)^N}\int (L^N\,\mathrm{e}^{\mathrm{i}k\psi})\phi\rho_0 f\,dx = \frac{1}{(\mathrm{i}k)^N}\int \mathrm{e}^{\mathrm{i}k\psi}(L^t)^N(\phi\rho_0 f)\,dx. \tag{5.50}$$

This clearly also satisfies the necessary bounds, so we have dealt with (5.48).

Finally we must estimate (5.49), for which we employ a similar method. We would like to write it, by definition of B, as

$$(2\pi)^{-n/2}\int dx \int \mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}\,\rho_0(x)\phi(x)b(\xi)\widehat{(\rho u)}(\xi)\,d\xi$$

but problematically, the integral in ξ may fail to converge. However, if we instead consider the restriction of $\widehat{(\rho u)}(\xi)$ to $|\xi| < R$, denoted by $\widehat{(\rho u)}_R(\xi)$, and exchange them in the above expression it then becomes valid. Letting L be defined by

$$Lw = \left|k\frac{\partial\psi}{\partial x} + \xi\right|^{-2}\left(k\frac{\bar{\partial}\psi}{\partial x} + \xi\right)\cdot\frac{\partial w}{\partial x}$$

we have $L^M(\mathrm{e}^{\mathrm{i}(x.\xi+k\psi)}) = \mathrm{i}^M\,\mathrm{e}^{\mathrm{i}(x.\xi+k\psi)}$ for all M and thus

$$(2\pi)^{-n/2}\int dx\int \mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}\,\rho_0(x)\phi(x)b(\xi)\widehat{(\rho u)}_R(\xi)\,d\xi = \mathrm{i}^M\int dx\int \mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}(L^t)^M(\rho_0\phi)b(\xi)\widehat{(\rho u)}_R(\xi)\,d\xi$$

for all R.

Since $\left|k\frac{\partial\psi}{\partial x}(x) + \xi\right|^2 > c(k^2 + |\xi|^2)$ for $x \in \operatorname{supp}(\rho)$ and $\xi \in \operatorname{supp}(b)$, we have

$$\left|(L^t)^M(\rho_0\phi)\right| \le C(k^2 + |\xi|^2)^{-M/2}$$

which means that we can bound $\mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}(L^t)^M(\rho_0\phi)b(\xi)\widehat{(\rho u)}_R(\xi)$ uniformly in R by an integrable function. Thus we may apply dominated convergence to deduce that (5.49) is equal to

$$\lim_{R\to\infty}\mathrm{i}^M\int dx\int \mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}(L^t)^M(\rho_0\phi)b(\xi)\widehat{(\rho u)}_R(\xi)\,d\xi = \mathrm{i}^M\int dx\int \mathrm{e}^{\mathrm{i}(\xi.x+k\psi)}(L^t)^M(\rho_0\phi)b(\xi)\widehat{(\rho u)}(\xi)\,d\xi,$$

which obeys the estimates (5.43) as required.

So we are left with the task of showing that the constants in (5.43) are uniform for (y,η) in $|y - y_0| < \delta_0, |\eta - \eta_0| < \delta_0$ if $\psi = \psi(x; y, \eta)$ such that $\frac{\partial\psi}{\partial x}(x_0; y_0, \eta_0) = \xi_0$ depends smoothly on parameters. We must also discuss how the proof can be modified to attain (5.39).

For the former, observe that all we need show is that we can find some $\delta_0 > 0$ such that whenever $|y - y_0| < \delta_0$ and $|\eta - \eta_0| < \delta_0$ we have:

- given $\delta > 0$ there exists $\delta' > 0$ such that $|x - x_0| < \delta'$ and $\left|\frac{\xi}{|\xi|} + \frac{\xi_0}{|\xi_0|}\right| > \frac{\delta}{2}$ imply that

$$\left|k\frac{\partial\psi}{\partial x}(x; y, \eta) + \xi\right| > c\left(k^2 + |\xi|^2\right)^{1/2}. \tag{5.51}$$

- given $\tilde{\delta} > 0$ there exists $r > 0$ such that $\mathrm{Im}(\psi(x; y, \eta)) > r > 0$ on the support of ϕ whenever $|x - x_0| > \tilde{\delta}$.

This will be sufficient, as these were the only conditions used in the proof.

To see that the first condition is possible, simply note that by continuity we can choose δ' small enough such that $|y - y_0| < \delta'$, $|\eta - \eta_0| < \delta'$ and $|x - x_0| < \delta'$ imply that $\frac{\partial \psi}{\partial x}(x; y, \eta)$ is sufficiently close to ξ_0 for (5.51) to hold for relevant ξ. The deduction of (5.51) given that $\frac{\partial \psi}{\partial x}(x; y, \eta) \sim \xi_0$ follows exactly as before.

For the second, assume that $\mathrm{Im}(\psi(x; y, \eta)) \geq c|x - x(y, \eta)|^2$ on $\mathrm{supp}(\phi)$ for some smooth $x(y, \eta)$ with $x(y_0, \eta_0) = x_0$. This means that we can find δ'' sufficiently small that $|y - y_0|, |\eta - \eta_0| < \delta'' \Rightarrow |x(y, \eta) - x_0| < \frac{\tilde{\delta}}{2}$. Then $|x - x_0| > \tilde{\delta} \Rightarrow |x - x(y, \eta)| > \frac{\tilde{\delta}}{2}$ and so $\mathrm{Im}(\psi(x; y, \eta)) > c(\frac{\tilde{\delta}}{2})^2 = r$ on $\mathrm{supp}(\phi)$.

Hence, combining the above and taking $\delta_0 = \min(\delta', \delta'')$ gives the required result.

Finally, to achieve (5.39) from the proof of the Theorem, we use a similar idea. Our goal is to show that given $t_0 \in (0, T)$ there exists some $\varepsilon > 0$ such that $\rho \in C_0^\infty(|t - t_0| < \varepsilon) \Rightarrow$

$$\left| \int_{[0,T]\times\Omega} \rho \bar{\omega} f \right| \leq C_N k^{-N}.$$

By the assumptions made on ω and f, we know that at the fixed time t_0, $(\bar{\omega} f)(t_0, \cdot)$ is of the correct form and satisfies the conditions needed to use the Lemma. That is, $(\bar{\omega} f)(t_0, \cdot)$ can be written as a sum (over all null bicharacteristics in the shower), with each term corresponding to some ψ, ϕ and (x_0, ξ_0) in the shower at time t_0, to which we can apply the Lemma. For simplicity we will consider only one term, the full result following trivially.

Take $\psi, \phi, (x_0, \xi_0)$ as fixed, corresponding to some null bicharacteristic at time t_0. Again by smooth dependence, this time of both ψ and f on t (before we only considered ψ), we can find $\varepsilon > 0$ small enough that whenever $|t - t_0| < \varepsilon$,

- given $\delta > 0$ there exists $\delta' > 0$ such that $|x - x_0| < \delta'$ and $\left| \frac{\xi}{|\xi|} + \frac{\xi_0}{|\xi_0|} \right| > \frac{\delta}{2}$ imply that
$$\left| k \frac{\partial \psi}{\partial x}(x; t) + \xi \right| > c \left(k^2 + |\xi|^2 \right)^{1/2}. \tag{5.52}$$
- given $\tilde{\delta} > 0$ there exists $r > 0$ such that $\mathrm{Im}(\psi(x; t)) > r > 0$ on the support of ϕ whenever $|x - x_0| > \tilde{\delta}$.

This means, by the same reasoning as our original proof, that for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$

$$\left| \int_\Omega (\bar{\omega} f)(t, x) \, dx \right| \leq C_N k^{-N}$$

for uniform constants C_N. Thus, for any $\rho \in C_0^\infty(|t - t_0| < \varepsilon)$

$$\begin{aligned}
\left| \int_{[0,T]\times\Omega} \rho \bar{\omega} f \right| &\leq \sup_{[0,T]} |\rho| \left| \int_{\mathrm{supp}(\rho)\times\Omega} \bar{\omega} f(t, x) \, dx dt \right| \\
&\leq \sup_{[0,T]} |\rho| \left| \int_{\mathrm{supp}(\rho)} \left(\int_\Omega \bar{\omega} f(t, x) \, dx \right) \, dt \right| \\
&\leq \sup_{[0,T]} |\rho| \times 2\varepsilon \times C_N k^{-N} \\
&\leq C_N k^{-N}
\end{aligned}$$

for some, alternative, constants $\{C_N : N \in \mathbb{N}\}$. □

References

[1] L. Hörmander: *Fourier Integral Operators I.* Acta Math., 79-183 (1971)

[2] J. Ralston: *Gaussian beams.* (2005)
Available at: http://www.math.ucla.edu/~ralston/pub/Gaussnotes.pdf (accessed 15/01/2014)

[3] J. Ralston: *Gaussian beams and the propagation of singularities.* Studies in Partial Differential Equations, MAA Studies in Mathematics 23, 206-248 (1983)

[4] W. Rudin: *Real and Complex Analysis.* McGraw-Hill, New York (1962)

[5] M. Taylor: *Partial Differential Equations I.* 2nd ed. Springer (2010)