On the topic of D-less numbers

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1 Introduction

The entire essay stems from a question which was on a maths challenge pack.

Find two whole numbers that multiply together to give one million, with added requirement that neither may contain the digit zero.

The answer is 2^6 and 5^6, else one of the numbers would have a multiple of 10 in and thus a zero digit. Also trivially, negative answers would work too, but we will only focus on the positive solutions. The answers agree: `http://sparx.co.uk/deck/easy/1.asp` This does however lead to the extension:

How many positive integer powers of 10 can be decomposed into two zeroless factors as described above?

Definition 1.1. ***D-less*** *A number N is D-less in base B, if it does not contain D in it's decimal expansion in base B. When base is not specified, I mean in base 10.*

This is a term I am coining, since I will be looking at various versions of D-less and I have only managed to find work on Zeroless numbers (that too, only in base 10.) e.g. 2, 22, 222, 2222, 22222... is a sequence of 3-less, 4-less... numbers in base 10. At this point, you may be thinking, well this isn't very difficult - but there is much more than meets the eye.

2 Powers

Claim. *There are infinite 0-less square numbers in base 10.*

Closely consider these two propositions, and we should be able to prove this claim. For $n \in \mathbb{N}$. [1] Let $x_n = 3\sum_{i=1}^{n} 10^i + 4$ for this section. This means $x_n = \underbrace{33\ldots3}_{\text{n times}}4$

Proposition 2.1.

$$6 \times x_n = 2 \times 10^{n+1} + 4 = 2\underbrace{0\ldots0}_{n\ times}4$$

[1] $\mathbb{N}$ = Naturals with 0. I will often be taking particular subsets of the naturals though.

Proof. This is done by induction. For n = 0, $x_0 = 4$. $4 \times 6 = 24 \quad \checkmark$.
Assume true for $n = k - 1 \in \mathbb{N}$
Consider $n = k$.

$$\begin{aligned} x_k &= 3 \times 10^k + x_{k-1} \\ x_k \times 6 &= 18 \times 10^k + 2 \times 10^k + 4 && \text{(By assumption)} \\ &= 20 \times 10^k + 4 \\ &= 2 \times 10^{k+1} + 4 \end{aligned}$$

As true for $n = 0$ and true for k when true for $k - 1$, true $\forall n \in \mathbb{N}$

$\square$

Proposition 2.2.

$$x_n^2 = \sum_{i=n+1}^{2n+2} 10^i + 5 \sum_{i=1}^{n} 10^i + 6 = \underbrace{1 \dots 1}_{n+1} \underbrace{5 \dots 5}_{n\ times} 6$$

Proof. This is done by induction. For n = 0, $x_0 = 4$. $4^2 = 16 \quad \checkmark$.
Assume true for $n = k - 1 \in \mathbb{N}$
Consider $n = k$.

$$\begin{aligned} x_k^2 &= (3 \times 10^k + x_{k-1})^2 \\ &= 9 \times 10^{2k} + 6 \times x_{k-1} \times 10^k + x_{k-1}^2 \\ &= 9 \times 10^{2k} + 2 \times 10^{2k} + 4 \times 10^k + x_{k-1}^2 && \text{(by prop above)} \\ &= 11 \times 10^{2k} + 4 \times 10^k + \sum_{i=k}^{2k} 10^i + 5 \sum_{i=1}^{n} 10^i + 6 && \text{(by assumption)} \\ &= \sum_{i=k+1}^{2k+2} 10^i + 5 \sum_{i=1}^{k+1} 10^i + 6 \\ &= \underbrace{1 \dots 1}_{\text{k+1}} \underbrace{5 \dots 5}_{\text{k times}} 6 \end{aligned}$$

As true for $n = 0$ and true for k when true for $k - 1$, true $\forall n \in \mathbb{N}$

$\square$

With this the earlier claim is also proved.

There is also a formula for a cubic 0-less sequence, but I feel this is more tedious than insightful. One of the things I wanted to investigate was if there is also a formula for higher powers. As far as I know, they are unfound, if they exist.

I wrote a script[2] which will output the numbers which are d-less in base 10 when taken to the 4^{th} power. Any sequence which does work should be a subsequence to this. If this makes any progress, I will then modify it to 5^{th} and 6^{th} powers...

I suppose the next logical step is to start looking at different bases too.

Claim. *There is only one 0-less square numbers in base 2.*

[2]test01.py

In base 2, numbers that are 0-less are of the form $1 \ldots 1$. Which are actually numbers of the form $2^n - 1 > 0$. So the claim is equivalent to saying there is only one numbers n s.t. $2^n - 1$ is a perfect square. We can quickly verify that 1 is both square and 0-less in base 2.

Proposition 2.3. *A perfect square mod 100 will be in*
$S = \{0, 1, 4, 9, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96\}$
$S = \{1, 4, 9, 16\} + 20\mathbb{N} \cup \{00, 25\}$

Proof. Square every number from 0 to 99 and take the last two digits. Observe that this gives each element from S. □

Proposition 2.4. $\nexists \in \mathbb{N}_{>1}$ *s.t.* $2^n - 1$ *is a perfect square.*

Proof. Take the powers of 2 higher that 2^1 modulo 100. They form into a cycle of size 20.

$$\begin{aligned} C &= \{4, 8, 16, 32, 64, 28, 56, 12, 24, 48, 96, 92, 84, 68, 36, 72, 44, 88, 76, 52\} \\ &= \langle < 52 > \rangle \\ &= \{20n + 4, 20n + 8, 20n + 12, 20n + 16 : n = 0, 1, 2, 3, 4\} \\ &= \{4, 8, 12, 16\} + 20\mathbb{N} \end{aligned}$$

Trivially, the set of $2^n - 1$ will be in the form A $= \{3, 7, 11, 15\} + 20\mathbb{N}$ we can easily see that it would be impossible for 0 or 25 to be in this set, and then can verify easily that the others are also not in the above set of perfect square residues mod 100. □

A second type proof:

Proof. Let n be even. $2^{2m} - 1 = q^2 \implies$ Two square numbers are 1 apart. But this is only true for 0 and 1. This would make $q^2 = 0$. But q^2 is the number we are testing to be 0-less. So no solutions.
Let n be odd. $2^{2m+1} - 1 = q^2$
$q^2 - 2(2^m)^2 = -1$ But this is a negative pell's equation. Solutions are obtained by $(x_0, y_0) = (1, 1)$ and $(x_{n+1}, y_{n+1}) = (3x_n + 4y_n, 2x_n + 3y_n)$.
Bearing in mind we need $y_n = 2^m$, we can consider the parity of the solutions. Other than the solution (1, 1) (i.e. $2^m = 1$ so $m = 0, 2^{2m+1} - 1 = 1$ and every other solution would be odd and so we have no more solutions. □

Thus also proving the claim. Trivially still, 0 is the only 1-less number in base 2. While we are searching other bases, it definitely does seem like hard work if we go through each base like this. But aha! a short-cut:

Proposition 2.5. *For base* $B \in \mathbb{N}_{>2}$*. There are infinite D-less power sequences for all powers for all* $D \in \mathbb{N}_{1<x<B}$

Proof. Consider the powers of B in base B. It will always be of the form $1\underbrace{0 \ldots 0}_{\text{n times}}$ by definition. Notice this will only ever use the 0 and 1 digits. So we can restrict all our work to those two as all else are trivial. □

In light of the last prop, I rewrote the program[3] earlier into two parts, each to go from all bases between 3 and 10 inclusive, only testing for 0-less and 1-less squares. The idea is that any sequence which does work, must be a subsequence of the one the program prints out. Should I ever think of a formula, I just need to look against this list.

Proposition 2.6. *$\nexists n \in \mathbb{N}_{>0}$ s.t. n^2 is 1-less in base 3.*

Proof. Consider x in base 3: $x = k * 3 + n$. $x^2 = 3 * 3k^2 + n^2$. For x^2 to be 1-less, $n^2 \not\equiv 1$. But $2^2 = 4 \equiv 1$, $1^2 = 1$. So final digit is 0. But if n was 0 then x can be divided by 3, and we get $\tilde{x} = x \div 3$. Eventually x must have a non-0 digit, as $x \in \mathbb{N}_{>0}$. That digit will produce a 1. □

Similarly,

Proposition 2.7. *$\nexists n \in \mathbb{N}_{>0}$ s.t. n^2 is 1-less in base 4.*

Proof. Consider x in base 4. $x = k * 4 + n$. $x^2 = 4 * (4k^2) + n^2$. For x^2 to be 1-less, $n^2 \not\equiv 1$. But $1^2 = 4 \equiv 1$, $3^2 = 9 \equiv 1$.
So final digit is 0 or 2. If n was 2, then notice. $x^2 = 16k^2 + 16k + 4$. So there is a '10' at the end of it's decimal expansion in base 4. As the rest is divisible by 16 (so in the 'hundreds' and higher columns). So it is still not 1-less. But if n was 0 then x can be divided by 4, and we get $\tilde{x} = x \div 4$. Eventually x must have a non-0 digit, as $x \in \mathbb{N}_{>0}$. That digit will produce a 1. □

MUCH MORE TO COME WHEN I GET AROUND TO IT. I even have the material, I just need to rigourise it and then ammend this. That said, I have been told that some of the proofs given are some what fundamental and not very pretty. If you find a better and not too lucrative proof, do email me.

Bibliography

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[3] test02.py