Hockey Stick Application

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I noticed a pattern which I thought was pretty neat. I showed it to my brother, who in turn showed it to my old maths teachers. They asked that I write up a short proof for it and show it off to the class. This is the same thing written up properly.

None of the mathematics in this paper is difficult. It's pretty much just A-level material.

1 Chapter 1 - A blunt introduction

Theorem 1.1. ***(The Identity in question:)*** $\forall n, k \in \mathbb{N}$

$$(k+2)\sum_{x=0}^{n}\prod_{i=0}^{k}(x+i) = \prod_{i=0}^{k+1}(n+i)$$

For the first few times you look at it, you may just think how elegant it looks. Or that's just me. Another thought that may occur is - But what do these symbols even mean!?

Well, let's just try and plug in a few numbers to see whether it's true.

Theorem 1.2. *(k = 0)*

$$2\sum_{x=0}^{n} x = n \times (n+1)$$

But hold on a second! That's Guass' formula! The notorious one he came up with at age 8 when asked what $1 + 2 + 3 + 4 + \cdots + 100$ was. If you can't see how this is true, just try the below proof by induction.

Proof. Let $n = 0$.

$LHS = 2 = 0\checkmark\, RHS = 0 * 1 = 0\checkmark$

So it's true for $n = 0$.

Now assume that the formula is true for $n = m \in \mathbb{N}$. Let's see what happens when we try plugging $m + 1$?

$$\begin{aligned} LHS &= 2\sum_{x=0}^{m+1} x \\ &= 2\sum_{x=0}^{m} x + 2(m+1) \\ &= m \times (m+1) + 2(m+1) \\ &= (m+2) \times (m+1) \\ &= RHS \end{aligned}$$

So, this is true for $n = 0$, and true for any natural number so long as the preceding number holds true. So by induction, this is true for all $n \in \mathbb{N}$ $\square$

So maybe now, you're thinking, well it worked for k = 0, surely it breaks at another k though? How do I know for sure this works?!

Well, we've proved one case by induction, so why stop?

Proof. Let $n = 0$. Fix $k \in \mathbb{N}$

$LHS = (k+2)\sum_{x=0}^{0}\prod_{i=0}^{k}(x+i) = 0$ as this has an empty sum $\checkmark$
$RHS = \prod_{i=0}^{k+1}(i) = 0 \times 1 \times 2 \times 3 \times \cdots \times (k+1) = 0\checkmark$

So it's true for $n = 0$.

Now assume that the formula is true for $n = m \in \mathbb{N}$. Let's see what happens when we try plugging $m + 1$?

$$
\begin{aligned}
LHS &= (k+2)\sum_{x=0}^{m+1}\prod_{i=0}^{k}(x+i) \\
&= (k+2)\sum_{x=0}^{m}\prod_{i=0}^{k}(x+i) + (k+2)\prod_{i=0}^{k}(m+1+i) \\
&= \prod_{i=0}^{k+1}(m+i) + (k+2)\prod_{i=0}^{k}(m+1+i) \\
&= m\prod_{i=0}^{k}(m+1+i) + (k+2)\prod_{i=0}^{k}(m+1+i) \\
&= (m+k+2)\prod_{i=0}^{k}(m+1+i) \\
&= \prod_{i=0}^{k+1}((m+1)+i) \\
&= RHS
\end{aligned}
$$

So, $\forall k \in \mathbb{N}$ this is true for $n = 0$, and true for any $m+1 \in \mathbb{N}$ as long as true for $m \in \mathbb{N}$. So by induction, this is true for all $n \in \mathbb{N}$. $\square$

But that proof seems a bit abstract, and the induction over double variable can be hard to understand sometimes.

2 Chapter 2 - A familiar face

Now, as mathematicians often do - I'm going to completely change topic. Firstly a few pretty neat results.

Theorem 2.1. ***(Factorial - Choose Identity)***
Let $n, k \in \mathbb{N}$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Sometimes this is given as a definition. But I like to think of it as a formula.

Proof. If you're trying to pick k unordered elements from n elements, then one way to go about this, is to pick them all in order, and then divide by all the ways we can make it. If we're picking k things, we have n choices for our first one, (n-1) for our second, ... and so we find $n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$ ways we can choose them in order.

But we don't want order! Just for argument's sake, let's say I do. How do I order k things? So if I have k things, then we have k choices for where to put the first one, (k-1) for the second, ... and so $k \times k-1 \times k-2 \times \cdots \times 1$.

So we have all the ways to choose k ordered things, and the amount of ways to order k things. But the ways to choose k ordered things would be the same as to choose them unordered and then order them.

$\binom{n}{k} \times k! = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$ $\square$

Theorem 2.2. ***Recursive choose formula***
Let $n, k \in \mathbb{N}$.

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Proof. If you're trying to pick k+1 elements from n+1 elements, then one way to go about this, is to pick a single element, say your least favourite one. Then choose k+1 from the remaining n. The only choices you haven't counted are the ones which include your least favourite element. But when it does, we're just picking k from the remaining n elements. So, after a bit of thought, this is easy. You could also be algebraic and prove this using the formula above. □

Corollary. $\forall n \in \mathbb{N}$

$$nchoose0 = 1$$

Proof. Only one way to choose 0 elements from anything - and that's do nothing! □

Did he just randomly introduce these topics for no reason other than to distract us? Yes, mostly.

3 Chapter 3 - Secrets of new friends

Let me introduce you to yet another new friend - Consider

$$\sum_{i=0}^{n} \binom{i+k+1}{i}$$

Now, let's consider it more.

$$\begin{aligned}
\sum_{i=0}^{n} \binom{i+k+1}{i} &= \binom{k+1}{0} + \binom{k+2}{1} + \cdots + \binom{k+1+n}{n} \\
&= \binom{k+2}{0} + \binom{k+2}{1} + \cdots + \binom{k+1+n}{n} \\
&= \binom{k+3}{1} + \cdots + \binom{k+1+n}{n}
\end{aligned}$$

That was just using the last two theorems from Ch2. I'm just going to reuse the second theorem a few times and eventually I find that:

$$\begin{aligned}
\sum_{i=0}^{n} \binom{i+k+1}{i} &= \binom{n+k+2}{n} \\
\sum_{i=0}^{n-1} \binom{i+k+1}{i} &= \binom{n+k+1}{n-1} \\
(k+2)! \sum_{i=0}^{n-1} \binom{i+k+1}{i} &= (k+2)! \binom{n+k+1}{n-1} \\
(k+2)! \sum_{i=0}^{n-1} \frac{(i+k+1)!}{(i)!(k+1)!} &= (k+2)! \times \frac{(n+k+1)!}{(n-1)!(k+2)!} \\
&= \frac{(n+k+1)!}{(n-1)!} \\
(k+2) \sum_{i=0}^{n-1} \frac{(i+k+1)!}{(i)!} &= \\
(k+2) \sum_{i=1}^{n} \frac{(i+k)!}{(i-1)!} &= \\
(k+2) \sum_{i=1}^{n} \prod_{x=0}^{k} (x+i) &= \prod_{x=0}^{k+1} (x+n)
\end{aligned}$$

Now doesn't that seem awfully familiar? The index on the sum is slightly different, but we can sort that out. $(k+2)\prod_{x=0}^{k}(x) = 0$ as above, but this is when we let the index $i = 0$. So adding it into the sum, will not change it's value.

Therefore:

$$(k+2) \sum_{i=0}^{n} \prod_{x=0}^{k} (x+i) = \prod_{x=0}^{k+1} (x+n)$$

That sure was a lot more involved, but it tells us a lot more about what the properties are.

When I get around to it, I will add some more chapters to this explaining how this is really cool.