Integer Partitions

Student ID: 1315929
Tutor: Roman Kotecky\Neil O'Connell

April 22, 2015

My study of Integer partitions came from a particular algebra assignment in which we were asked to find all the isomorphism types for abelian groups of order 800. So I wondered:
Given any integer n, how many isomorphism types are there?

Theorem 0.1. ***(Fundamental theorem of finitely generated abelian groups.)*** *Let G be a finitely generated group. Then $\exists$ integers r,k and $d_1, \ldots, d_r \geq 2$ with $d_i | d_{i+1}$, $\forall i$ s.t.*

$$G \cong (\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}) \oplus \mathbb{Z}^k$$

(For $k = r = 0, G = 0$).

Now, $k = 0 \iff$ G is finite, and considering this is what my question concerns, we can assume $k = 0$. There are several ways we can write down a positive integer. E.g. $2015 = 2014 + 1 = 2013 + 2 = 2013 + 1 + 1 \ldots$ instead of finding all of the ways, I really want to know how many different ways I can write a positive integer n.

So, if we consider my question again, I am asking, given a number n, how many different d_i exist, s.t. $d_1, \ldots, d_r \geq 2$ and $d_i | d_{i+1}$, $\forall i$ and $\prod_{i=0}^{r} d_i = n$

Now, I have a formula that is fairly easy to prove, but first a few definitions.

Definition 0.2. ***(Integer Partitions)***

$$p(n,k) = \textit{ the number of ordered partitions of n into exactly k parts}$$

Clearly, we can also define/see this:

Theorem 0.3.

$$p(n) = \textit{ the number of ordered partitions of n with addition} = \sum_{k=1}^{n} p(n,k)$$

This is true as the number of ways I can partition a number at all, would be the number of ways I partition it into 1 or 2 or ... n and so on, if $k > n$ then $P(n,k) = 0$. And voila:

Theorem 0.4. ***(Number of isomorphism types)*** *Writing $n = p_1^{\alpha_1} \ldots p_q^{\alpha_q}$*

$$I(n) = \textit{ the number of isomorphism types of abelian groups order } n = \prod_{i=1}^{q} p(\alpha_i)$$

Proof. The total amount of ways we can make such d_i from above, would be to consider all the different numbers of the form $p_1^{\beta_{i_1}} \dots p_q^{\beta_{i_q}}$ for each d_i, since we want $\prod_{i=0}^{r} d_i = n$. Also, we need $d_i | d_{i+1}$, so we would need $\forall i \in [1, r]$, $\beta_{i_j} \leq \beta_{i+1_j}$. s.t. $\sum_{i=0}^{r} \beta_{i_j} = \alpha_i$. (the sum of the powers of a particular prime for all the d_i, is equal to the power of the same prime in n.
For the above, remember i relates the different d_i, j is per prime, $(i, j) \in [1, r] \times [1, q]$.
Now, for each prime, we want all the different ways to write q numbers as an ordered integer partition. ($\sum_{i=0}^{r} \beta_{i_j} = \alpha_i \implies$ integer partition. $\beta_{i_j} \leq \beta_{i+1_j} \implies$ ordered.)
So, for each prime, we have $I(p_i^{\alpha_i}) = p(\alpha_i)$.
But, for a composite number, we would have to partition each α_i, in a different way, to get all combinations. As each partition is independent, (primes are independent), we can see $I(n) = \prod_{i=1}^{q} p(\alpha_i)$. □

Corollary. *An immediate corollary is that I(n) is multiplicative.*
If $gcd(a, b) = 1$, then $I(a)I(b) = I(ab)$
This is also proven in my last sentence of main proof. □

This is how I came to find integer partitions, and an useful application of them. Now usually, combinatorial functions such as $p(n)$ will have fairly strange formulae with factorials here and there, but nothing too complicated. So I search up the formula, and I am confronted with:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\, A_k(n)\, \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh \left[\frac{\pi}{k} \sqrt{\frac{2}{3} \left(n - \frac{1}{24} \right)} \right] \right).$$

The $A_k(n)$ function is another strange function involving the Dedkind sum.
I don't even know how a function with so many transcendentals and non-integer functions manages to always spit out an integer, but I do want to know. As well as briefly looking at the formula, we will look at the applications of integer partitions (often where the real beauty in combinatorics lies).

1 Integer Partitions

There does exist a formula, thanks to Hardy and Ramanujan and J. V. Uspensky, (and countable others before them), but the formula is not only too intricate, but far beyond my scope to study it.

Instead we will be looking at applications of $p(n)$, a much more interesting (and my level) piece of work.

One may also want to ask, what if I want all the partitions with a given basis, (not the complete integers). For example, what are all the ways of having £2.50 with different coins? This would lead us on to generating functions.Whenever G.F.s are mentioned, I am assuming that $|x| < 1$.

Theorem 1.1. ***(Partitions into basis of choice):*** *The generating function that gives you partitions in a certain basis is:*

$$G(x) := \prod_{i \in B} \frac{1}{1 - x^i}$$

where B is the basis set.

Proof. For every way above that x^n is made its coefficient counts by 1. So the coefficient of x^n will be the amount of ways to partition n with the chosen basis, in this case the value of coins. "k" represents the amount of that coin that we are using, the values of the basis being the powers in this formula. □

Corollary. ***(Change making GF):*** *The generating function for making change in the british monetary system would be:*

$$C(x) := \prod_{i \in B} \frac{1}{1 - x^i}$$

$B = \{1, 2, 5, 10, 20, 50, 100, 200\}$ □

Example:
x^3 can be made with (all in pence), 1 + 1 + 1 = 2 + 1 , so we expect the coefficient to be 2. x^5 can be made with: 5 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 , so we expect the coefficient of x^5 to be 4. So, let's multiply out and see:

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^5} + Ox^6$$

$$(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4)(1 + x^5) + Ox^6$$

$$(1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4 + x^5) + Ox^6$$

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 4x^5 + Ox^6$$

And the coefficient of x^5 certainly is 4, we can also verify x^3 from above.

Corollary. ***(Integer partitions generating function):***

$$\sum_{k=0}^{\infty} p(k)x^k := \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} (x^{ik})$$

□

If we forget ordering for a while, we can obtain an upper bound for each p(n — k) . For p(n — k parts, ordered) each x_i can contribute 1,2,3 ... to the total sum. So generating function for each x_i is $x + x^2 + x^3 + \dots$
so $(x + x^2 + x^3 + \dots)^k = x^k/(1 - x)^k = (1/(1 - x) - 1)^k$ is the G.F. for p(n — k parts, ordered). This is fairly easy to calculate: (n-1, k-1), Of course this is if we care about the order, but gives us an upper bound. This isn't quite as helpful as an actual easy to calculate formula.

Something else we could ask, what if we want only partitions using 1 and 2, from

With the generating function, Euler noticed his "pentagonal number theorem". That and the GF imply that $p(n) = \sum_{k \neq 0} p(n - g_k)$ Where g_k are the generalised pentagonal numbers.

2 Pentagonal Number Theorem

Theorem 2.1. ***(Pentagonal Number Theorem)***

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{-\infty}^{\infty}(-1)^k x^{g_k}$$

Where $g_k=\frac{k(3k+1)}{2}$ *and* $|x|<1$.

Proof can be found in psu taba book page 45-47.
The proof works on the idea that the there is a map between partitions of even length and those of odd length, usually a bijection (thus all the cancellations). But differ for pentagonal numbers, in which case they stay with the differing signs. Another proof can be found by Dyson, which also uses partitions but looks at it from a different angle. The rank of a partition is defined and this is then used to create some recursions and then a similar bijection. This is perhaps harder to follow. This relation may seem quite abstract but helps give a recurrence relation for $p(n)$ which helps us compute it for big numbers. The derivation is mostly manipulation of generating functions.

With the pentagonal number theorem and the G.F. above:

$$\frac{1}{\prod_{n=1}^{\infty}(1-x^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-x^n)}=\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}x^{kn}$$

$$\left(\sum_{k=0}^{\infty}p(k)x^k\right)\left(\sum_{-\infty}^{\infty}(-1)^k x^{g_k}\right)=1$$

Now we have a polynomial on the left and on the right. So if we equate co-eff's for a general x^n, we see:

$$p(n)=\sum_{k\neq 0}^{\infty}(p(n-g_k))$$

or so that we can see it easier:

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+\ldots$$

Creating a finite recursive relationship 19th and 29th century mathematicians could use to work out the values of p(n). And as usual, there are plenty of recurrence relations. But this is the most useful one used by mathematicians.

3 Recurrences

Another way we can find $p(n)$ is with recurrence relationships. Note the following: $p(1,1)=1$, $p(n,k)=0\forall k\notin I_1^n$. These two are simple enough to see, and another slightly more difficult:

Theorem 3.1. $n\geq 2$ *and* $k\in I_1^n$

$$p(n,k)=p(n-1,k-1)+p(n-k,k)$$

Proof can be found in first book p110.

Using this recurrences and the two facts above, we can work out any p(n,k) and thus also any p(n) is we work out enough iterations. Thankfully, we can solve some recurrence relations to make this easier for us.

Insert example of solved reccurence - talk about the others on wolframalpha's page.

Using the formula before this chapter:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

it was very easy to make a log book of all the values to a fairly high number. Several years later, a relatively new mathematician was shown these, and he instantly conjectured and proved something known forever more as the "Ramanujan Congruences".

4 Congruences

Ramanujan discovered that

$$p(5n+4) \equiv 0(mod 5)$$

and

$$p(7n+5) \equiv 0(mod 7)$$

$$p(11n+6) \equiv 0(mod 11)$$

When looking at the table of value it's fairly easy to see this, the proof I found revolves entirely about rearranging polynomials.

So look at (one version of) the Jacobi Identity.

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+\frac{x^{2n-1}}{z^2}) = \sum_{m=-\infty}^{\infty} x^{m^2}z^{2m}$$

Just by several lines of rearranging, and case analysis for the coefficients mod 5, one can prove the first one. That said, it is still imensely difficult. Ramunajan also conjectured a more general congruence relation, but this turned out to be false for various powers of 7.

Conjecture: If $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1(mod\delta)$, then $p(\delta n + \lambda) \equiv 0(mod\delta)$
In truth, he was not far off:

Theorem 4.1. ***Congruences***
If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1(mod\delta)$, *then* $p(\delta n + \lambda) \equiv 0(mod 5^a 7^{\lfloor \frac{b}{2} \rfloor + 1} 11^c)$

Now, this section was completely out of my depth, I understand what each theorem says, but cannot by any means understand the proofs. Ramanujan did work on several parts of partiton theory, his congruence, the assympototic formula given in chapter 0, and also something which was known, but derived independently, Roger's theorem.

Theorem 4.2. ***(Roger's Theorem)*** *The number of partitions whose parts differ by at least 2, is equal to the number of partitions involving numbers congruent to* $\pm 1 (mod 5)$.

Some other interesting results that I will not prove:

Theorem 4.3. ***(Glaisher's Theorem)*** *When partitioning, the number of partitions with parts not divisible by d is the same as the number of partitions such that no part is repeated d or more times.*

When d = 2, this becomes Euler's Theorem.

Corollary. ***(Euler's Theorem)*** *The number of partitions into odd numbers is same as number of partitions into distinct numbers* □

Proof. From the Generating Function earlier, we use a similar argument. The GF of partitioning into odd numbers would be:

$$\prod_{i \in 2\mathbb{N}+1} \frac{1}{1-x^i}$$

And the GF of partitioning into distinct parts would be:

$$\prod_{i \in \mathbb{N}} 1 + x^i$$

Now, note: Partitions into distinct numbers

$$\begin{aligned} \prod_{i \in \mathbb{N}} 1 + x^i &= (1+x)(1+x^2)(1+x^3)\dots \\ &= \frac{(1-x^2)(1-x^4)(1-x^6)\dots}{(1-x)(1-x^2)(1-x^3)\dots} \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\dots} \end{aligned}$$

= number of partitions into odd parts

□

5 Construction of the formula

Now if we remember the formula we saw earlier, which was very complex and found by Hardy and Ramanujan, was initially just assympototic. But we can solve this for some simplified cases, these results should confirm the Theorem above when partitioning over a finite basis. None of this chapter is from a book or source, this is me engaging with the content, and seeing if things corroborate.

Firstly, if we let p_n to be partition of n into 1's and a's for some fixed $a > 1$ in naturals. Now, $p_n = 1 + p_{n-a}$.

Proof: As in any partition, writing the partition are in ascending order, last number is 1 or a. If last number is 1, then all the numbers previously would be 1. So, in total 1 partition. If last number is a, then we can construct a natural bijection between these partitions and ones for p_{n-a}.

Now, this would mean that if we write $n = q \times a + r$, recursively, $p_n = q + p_r$. But $r < a$, so only partition of r would be entirely in 1's. so $p_n = q + a$.

The beauty of this result is that it that $1|a$. If we want to extend this result to general $a, b \in \mathbb{N}$, we have already solved the case for when $a|b$! The only numbers partitionable when partitioning into such a and b, would be multiples of a, else 0. But then we notice, that this would just be a multiple of problem of partitioning $\frac{n}{a}$ into 1's and $\frac{b}{a}$. Something which we just solved.

So now, it rests upon when $a \nmid b$, WLOG $a < b$. Using a similar approach to above, we get to $p_n = p_{n-b} + (1 \text{ if } a|n)$ as in any partition, last number is a or b. If last one is a, then all numbers before must have been a too, but this is only the case when $a|n$. If last one is b, then we can construct a natural bijection between these and ones for p_{n-b}. Solving this recursively was not as straight forward, things get messy and go into several cases. So, instead I built a Java file which took an input and then returned the number of partitions. It also calculated the coefficients of the generating series and then showed that they are equal. File attached at end.

Maybe end on section(hard stuff that i can't actually do.)

Maybe also Durfee square and also the ferreer's diagrams – more my level, but doesn't seem that interesting.

Definition 5.1. ***(Insert relevant Definition)***

$$Insert\,Definition$$

Theorem 5.2. ***(Insert relevant name)***

$$Insert\,Theorem$$

Proof. Proof □

Corollary. *Cor* □

Method. *Method*

Bibliography

Pearls of discrete mathematics – Martin Erickson chapter 15
Daan's Algebra 1 Lecture Notes.
http://mathworld.wolfram.com/PartitionFunctionP.html
http://www.artofproblemsolving.com/Resources/Papers/LaurendiPartitions.pdf
https://oeis.org/A000041
http://www.ams.org/notices/200109/fea-ahlgren.pdf
http://web.maths.unsw.edu.au/ mikeh/webpapers/paper59.pdf
http://mathworld.wolfram.com/PartitionFunctionPCongruences.html
http://www.sciencedirect.com/science/article/pii/S0021980069800062
http://www.math.psu.edu/tabachni/Books/taba.pdf
http://www.mathcs.emory.edu/ ono/publications-cv/pdfs/017.pdf
http://math.stackexchange.com/questions/54961/the-number-of-partitions-of-n-into-distinct-parts-equals-the-number-of-partiti