Fermi acceleration in non-autonomous billiards

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Abstract

Fermi acceleration can be modelled by a classical particle moving inside a time-dependent domain and elastically reflecting from its boundary. In this paper we describe how the results from the Dynamical System theory can be used to explain the existence of trajectories with unbounded energy. In particular, we show for slowly oscillating boundaries that the energy of the particle may increase exponentially fast in time.

In paper [1] Fermi noticed that a particle's energy can grow due to repetitive interactions with an oscillating potential wall (for a recent discussion see [2, 3]). In the simplest (Fermi-Ulam) model the particle bounces elastically between two parallel walls, one of which is moving and the other one is fixed. The particle gains energy in head-on collisions and looses it in overtaking ones. If oscillations of the moving wall are random, the amounts of kinetic energy gained (or lost) by the particle in consecutive collisions are not correlated and, on average, are positive. Therefore the particle typically accelerates (e.g. see the discussion in [4]). If the motion of the wall is deterministic, for example periodic in time, the situation is different [5]. When the particle moves rapidly the velocity of the wall does not notably change between two consecutive collisions. Therefore the dynamics of the particle exhibits alternate periods of acceleration and deceleration, and over long time the particle energy oscillates periodically due to the presence of adiabatic invariants [6]. Moreover, the periodic in time Fermi-Ulam model can be interpreted as a Hamiltonian system with two degrees of freedom and therefore invariant (KAM) curves may present permanent barriers for the Fermi acceleration [5].

A more sophisticated model is provided by a particle inside a billiard with a time-dependent boundary. The behaviour of the particle depends both on the

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shape of the billiard domain and on the way it changes [7]. For a fast particle the boundary of the billiard moves slowly and the billiard with the frozen boundary provides a good first order approximation for the particle motion. If the frozen billiard is dispersing or of any other shape that creates a chaotic particle motion, the correlations decay fast and the similarity to the case of the random boundary motion seems to be obvious.

A more accurate approximation is provided by the averaging theory which predicts existence of adiabatic invariants. Adiabatic invariants are functions on the phase space which keep their values nearly constant for a long time, so their existence represents an obstacle to a systematic growth of the energy. The adiabatic invariants exist in two quite different situations: either when the billiard with the frozen boundary is integrable or when it is completely chaotic (see for example [9, 8]). Of course, adiabatic invariants are preserved only approximately and do not prevent Fermi acceleration on a long time scale. It is important that the preservation of the adiabatic invariant in the chaotic case holds for a much shorter time scale than in the integrable case. We also note that in both cases adiabatic theory does not apply to all initial conditions.

In an integrable system adiabatic invariants can be destroyed due to separatrix crossings. In [10] Itin and Neishtadt applied the general theory to study the destruction of adiabatic invariants near a separatrix in an elliptic billiard with slowly changing parameters. Recently Lenz *et al.* [11] numerically detected Fermi acceleration for particles with initial conditions in this region and reported that corresponding acceleration rates are much slower compared to non-integrable billiards.

If the frozen billiard is chaotic, the averaging theory applies only to initial conditions which have good "ergodization" properties, *i.e.*, to those for which a time average converges to the space average in a controllable way [12]. For the rest of the initial conditions the behaviour may deviate strongly from the average. In particular, in [13] we rigorously proved that in a smooth Hamiltonian system a chaotic behaviour will produce orbits of unbounded energy when a slow time-dependent perturbation of the Hamiltonian function is applied. The present paper explains an application of our theory to the problem of Fermi acceleration in non-autonomous billiards.

Consider a particle of mass m that travels inside a time-dependent domain elastically reflecting from its boundary. Between collisions the particle moves with a constant speed along straight lines connecting collision points. Let $\mathbf{v}$ denote the velocity of the particle, then its total energy is given by

$$h = \frac{m|\mathbf{v}|^2}{2}\,.$$

The energy of the particle may change due to a collision with the moving boundary and is not preserved. In this paper we describe a mechanism which leads to unbounded growth of the energy for some of the initial conditions.

If the particle velocity is sufficiently large in comparison with the speed of the boundary motion, then the time between consecutive collisions is small and we can assume that the shape of the billiard does not change substantially during the time required for several consecutive collisions. Moreover, the relative change in the particle speed due to a single collision is small. Then we can fix time t and consider the corresponding billiard with a frozen boundary as a zero order approximation for the particle motion.

Suppose that the frozen billiard has a closed orbit L of length ℓ. It is convenient to define the action of the periodic orbit by

$$J = \oint_L \mathbf{p}\, d\mathbf{q}\,.$$

Taking into account that $\mathbf{p} = m\mathbf{v}$ and $\mathbf{v}$ is constant and parallel to the orbit on each of the orbit's segments, we can evaluate this integral exactly:

$$J = J(h,t) = m|\mathbf{v}|\ell(t) = \sqrt{2mh}\,\ell(t)\,. \tag{1}$$

The period of the frozen periodic orbit, $T = T(h,t)$, can be expressed in terms of the action:

$$T(h,t) = \frac{\ell(t)}{|\mathbf{v}|} = \sqrt{\frac{m}{2h}}\ell(t) = \frac{\partial J}{\partial h}\,. \tag{2}$$

We note that periodic orbits of the frozen billiard form two-parameter families: one parameter is the frozen time t and the second one is the energy h, since the particle can go around the polygon L at different speeds. We denote this collection of periodic orbits by $L(t,h)$.

Now let us consider a trajectory of the non-autonomous billiard which has initial conditions close to a periodic orbit on $L(t,h)$. This trajectory stays near the periodic orbit for some time. After making one rotation the energy changes:

$$\Delta h := h(t+T) - h(t) \approx -\frac{\partial J}{\partial t} = -\sqrt{2mh}\,\dot{\ell}(t)\,. \tag{3}$$

This formula follows from the general theory of adiabatic invariants in slowly driven Hamiltonian systems (see more details in [13]). We provide an elementary derivation of this formula directly from the elastic reflection law.

Suppose the periodic orbit $L(t)$ hits the boundary of the billiard at the sequence of points $\mathbf{z}_j(t)$, $j = 1,\ldots,n_0$. We let $\mathbf{z}_0 = \mathbf{z}_{n_0}$ and define $\mathbf{r}_j = \mathbf{z}_j - \mathbf{z}_{j-1}$, $j = 1,\ldots,n_0$. The vectors $\mathbf{r}_j$ represent the straight-line segments which form the periodic orbit. Let $\mathbf{n}_j$ denote the internal unit normal to the boundary at $\mathbf{z}_j$. The normal velocity of the boundary at the j-th collision point is given by $u_j := \dot{\mathbf{z}}_j \cdot \mathbf{n}_j$. By assumption, u_j is small.

Since the particle starts near $L(t)$, it makes one round hitting the boundary near the points $\mathbf{z}_j$ consecutively. Since the reflections are elastic, each collision leads to the following change in the energy:

$$\Delta E_j = 2m(-u_j \mathbf{v}_j \cdot \mathbf{n}_j + u_j^2)\,, \tag{4}$$

where $\mathbf{v}_j$ is the velocity of the particle before the collision near $\mathbf{z}_j$; since the boundary moves slowly compared to the particle, the change in the energy is relatively small and in the leading order we can write $\mathbf{v}_j = |\mathbf{v}|\mathbf{r}_j/|\mathbf{r}_j|$. By neglecting the square of u_j, we get

$$\Delta h = \sum_{j=1}^{n_0} \Delta E_j = -2m|\mathbf{v}| \sum_{j=1}^{n_0} u_j \frac{\mathbf{r}_j \cdot \mathbf{n}_j}{|\mathbf{r}_j|} \,. \tag{5}$$

On the other hand, we have $J = m|\mathbf{v}| \sum_{j=1}^{n_0} |\mathbf{r}_j|$ by (1), so

$$\frac{\partial J}{\partial t} = m|\mathbf{v}| \sum_{j=1}^{n_0} \frac{\mathbf{r}_j \cdot \dot{\mathbf{r}}_j}{|\mathbf{r}_j|} = m|\mathbf{v}| \sum_{j=1}^{n_0} \frac{\mathbf{r}_j \cdot (\dot{\mathbf{z}}_j - \dot{\mathbf{z}}_{j-1})}{|\mathbf{r}_j|} \,.$$

In this sum each of $\dot{\mathbf{z}}_j$ appears twice, and taking into account the periodicity of the orbit we can reorder the sum:

$$\frac{\partial J}{\partial t} = m|\mathbf{v}| \sum_{j=1}^{n_0} \left(\frac{\mathbf{r}_j}{|\mathbf{r}_j|} - \frac{\mathbf{r}_{j+1}}{|\mathbf{r}_{j+1}|} \right) \cdot \dot{\mathbf{z}}_j \,.$$

In the frozen billiard the angle of incidence is equal to the angle of reflection, therefore the tangent component of the difference cancels while the normal component doubles:

$$\frac{\partial J}{\partial t} = 2m|\mathbf{v}| \sum_{j=1}^{n_0} \frac{\mathbf{r}_j \cdot \mathbf{n}_j}{|\mathbf{r}_j|} \, \mathbf{n}_j \cdot \dot{\mathbf{z}}_j = 2m|\mathbf{v}| \sum_{j=1}^{n_0} u_j \frac{\mathbf{r}_j \cdot \mathbf{n}_j}{|\mathbf{r}_j|} \,.$$

Comparing this with (5) we obtain the desired identity (3).

By (3),(2), the energy of the particle closely follows a solution of the differential equation

$$\dot{h} = -\frac{1}{T} \frac{\partial J}{\partial t} = -2h \frac{\dot{\ell}}{\ell} \,. \tag{6}$$

We see that the time derivative of the energy is not small. On the other hand, the full time derivative

$$\dot{J}(t, h(t)) = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial h} \dot{h}$$

vanishes along solutions of the differential equations (6) due to (2). Therefore if the particle stays close to $L(t)$, the action J is an adiabatic invariant, i.e., the energy of the particle changes in such a way that $J(h,t)$ remains approximately constant during a long period of time.

Equation (6) can be easily integrated:

$$h(t) = h(0) \, \frac{\ell^2(0)}{\ell^2(t)} \,. \tag{7}$$

We see that the energy stays bounded unless the length of the periodic orbit vanishes. It is convenient to let $w = -2\dot{\ell}/\ell$, then equation (6) takes the form

$$h^{-1} \dot{h} = w(t) \,. \tag{8}$$

We note that if the trajectories on $L(h,t)$ are hyperbolic then the theory of normal hyperbolicity [14] implies that there are trajectories of the non-autonomous billiard which stay near $L(h,t)$ with $h \gg 1$ for very long times.

It is seen from (7) that a trajectory, which stays near L, does not accelerate. The possibility of accelerating trajectories depends on the dynamics of the frozen billiard. Suppose the frozen billiard has chaotic dynamics at least in a part of the phase space. In other words, the frozen billiard has a Smale horseshoe. This can be described by the existence of a couple of hyperbolic periodic orbits, L_a and L_b, connected by a couple of transversal heteroclinics (there are infinitely many such pairs within the chaotic set, we just choose any two of them). The dynamics near the horseshoe can be described using symbolic dynamics: for any sequence composed of symbols a and b, there is a trajectory which switches between L_a and L_b following the order prescribed by the sequence.

In [13] we used normal hyperbolicity arguments in the spirit of [15, 14, 16] in order to show that this property is inherited by slowly changing systems, which includes as a special case the billiard map under quite general assumptions on its boundary. We proved that in the presence of the Smale horseshoe the slow non-autonomous billiard with a fast particle inside has trajectories which switch between two small neighbourhoods of L_a and L_b in an arbitrary, prescribed in advance order. This freedom can be used to achieve an optimal strategy for the acceleration. Let us study the behaviour of a trajectory which stays near L_a if $w_a(t) > w_b(t)$ and stays near L_b otherwise (we assume that the changes of the billiard shape are not very large, so the periodic orbits L_a and L_b persist in the frozen billiard for all t). The energy of the particle changes in time approximately as a solution of the differential equation (8) with $w = w_a$ or $w = w_b$ depending on time t:

$$h^{-1}\dot{h} = W(t) := \max\{\, w_a(t), w_b(t)\,\}\,.$$

This equation has an obvious solution:

$$h(t) = h(0)\exp\int_0^t W(t)\,dt\,. \tag{9}$$

We note that under very general assumptions these solutions are not bounded. For example, if the oscillations of the boundary are periodic, then ℓ_a and ℓ_b are periodic functions of time. Then $w_{a,b}$ are derivatives of periodic functions and therefore have zero mean: $\overline{w}_{a,b} = 0$. Since W is the maximum of two periodic functions with zero mean, the mean $\overline{W}$ is positive provided w_a and w_b do not coincide over the entire period or, equivalently, provided $\ell_a(t)/\ell_b(t)$ is not constant. Then we conclude from (9) that the energy of the particle oscillates around the exponentially growing function:

$$h(t) \approx h(0)\exp\overline{W}t\,. \tag{10}$$

In the arguments above, the periodicity assumption substantially simplified the analysis. In fact, the periodicity is not an essential part of the arguments:

the only important things are that the billiard shape is changing (i.e. ℓ_a/ℓ_b varies with time), that the boundary motion is slow and that the phase space of the frozen billiard retains the Smale horseshoe for all times. Then, both periodic and non-periodic cases can be treated by the proposed method, see the corresponding theory in [13, 17].

We note that the trajectories constructed above have the largest acceleration. For an arbitrary trajectory from the vicinity of the horseshoe, we find

$$h^{-1}\dot{h} \approx W_\xi(t)\,dt\,, \tag{11}$$

where the sequence ξ of a's and b's is the code that describes the switches of the orbit between the neighbourhoods of L_a and L_b, and $W_\xi(t)$ is a function obtained by the concatenation of the segments $w_a(t)$ and $w_b(t)$ in the order given by the code ξ. We may consider ξ as being random, so $W_\xi(t)$ is a random process. Its mean value is given by $\overline{W}_\xi(t) = \lambda w_a + (1-\lambda)w_b$ for some $\lambda \in (0,1)$. For a typical ξ the values of $W_\xi(t)$ and $W_\xi(t+\tau)$ are uncorrelated at $\tau > \max\{\,T_a, T_b\,\} \sim h^{-1/2}$. Therefore, when h is sufficiently large, we may view $W_\xi(t) - \overline{W}_\xi$ as a white noise of power $(1-\lambda)^2(w_a-w_b)^2h^{-1/2}$. Thus, equation (11) can be modelled by

$$\dot{h} = (\lambda w_a + (1-\lambda)w_b)h + h^{3/4}(1-\lambda)|w_a - w_b|\eta(t),$$

where η is the white noise of power 1. We take $\lambda = 1/2$ for simplicity and, as $w_{a,b} = -2\dot{\ell}_{a,b}/\ell_{a,b}$, we find

$$h(t)\ell_a(t)\ell_b(t) = \left[(h(0)\ell_a(0)\ell_b(0))^{1/4} + \frac{1}{4}\int_0^t (\ell_a(s)\ell_b(s))^{1/4}\left|\frac{d}{ds}\ln\frac{\ell_a(s)}{\ell_b(s)}\right|\eta(s)ds\right]^4.$$

As $\ell_{a,b}$ remain bounded away from zero and infinity, we may write

$$h(t) \sim u^4G(t)^4,$$

where $u \sim \dot{\ell}_{a,b}$ is the average velocity of the change in the boundary shape and $G(t) \sim \int_0^t \eta(s)ds$ is a Gaussian random variable with zero mean and dispersion t. Hence,

$$<h(t)> \sim u^4t^2. \tag{12}$$

One can check that the same conclusion holds true in the vicinity of an arbitrary uniformly-hyperbolic set of the frozen billiard. For the dispersing Sinai billiards (e.g. Lorenz gas) the uniformly hyperbolic sets exhaust all the phase space, so in this case the quadratic energy growth should be a typical phenomenon.

In terms of the number of collisions n, formula (12) transforms into

$$<h(n)> \sim u^2n. \tag{13}$$

Indeed, by (12), $|\mathbf{v}| \sim \sqrt{h} \sim u^2t$. As the density Δn of the number of collisions in the interval Δt is estimated as $\Delta n \sim |\mathbf{v}|\Delta t$, we find $dh/dn \sim |\mathbf{v}|^{-1}dh/dt \sim u^2$, which gives (13) indeed. As we see, the effective energy gain for one collision is of order u^2, *i.e.*, it is of the same order as the terms we neglected in the derivation of Eq. (3) that describes the evolution of h to the first order in u. It follows that a more accurate analysis of the typical motion, which takes into account the terms of the order of u^2, would lead to the same energy growth laws (13), (12). We note that (13) is in a good agreement with the results of numerical experiments in [2, 7].

We have found that non-autonomous non-integrable billiards have orbits of exponential energy growth. The effect persists for any billiard of a slowly changing shape provided the frozen billiard system possesses a region of chaotic dynamics in the phase space. On average the energy growth is much slower: for a typical orbit from the vicinity of the chaotic set the energy growth rate is quadratic in time, like in the classical Fermi acceleration.

Finally we point out that our arguments do not require the billiard domain to be two-dimensional.

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