Unique resonant normal forms for area preserving maps at an elliptic fixed point‡

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Abstract. We study possible simplifications of normal forms for area-preserving maps near a resonant elliptic fixed point. In the generic case we prove that at all orders the Takens normal form vector field can be transformed to the particularly simple form described by the formal interpolating Hamiltonian

$$H = I^2 A(I) + I^{n/2} B(I) \cos n\varphi \,.$$

The form of formal series A and B depends on the order of the resonance n. For each $n \geq 3$ we establish which terms of the series can be eliminated by a canonical substitution and derive a unique normal form which provides a full set of formal invariants with respect to canonical changes of coordinates.

We extend these results onto families of area-preserving maps. Then the formal interpolating Hamiltonian takes a form similar to the case of an individual map but involves formal power series in action I and the parameter.

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1. Introduction

Normal forms provide an important tool for the study of dynamical systems (see e.g. [1, 9, 17, 7]) as they can be used to achieve a substantial simplification of local dynamics. The normal form theory uses changes of variables to transform a map or a differential equation into a simpler one called a normal form. Usually the normal form is more symmetric than the original system and sometimes (but not always) allows an explicit study of the local dynamics. Classical normal forms are not always unique and their further reduction has been studied by various methods in order to derive unique normal forms [2, 3, 16, 10].

We study possible simplifications of normal forms for area-preserving maps near a generic resonant elliptic fixed point. The normal forms are obtained by simplifying the Takens normal form vector fields first introduced in [20]. The leading order of the normal form is well known and can be found in the classical book [1]. The complete formal description of our normal forms is provided by Theorems 1.3 and 1.4.

Our proofs use the classical strategy and are inductive in their nature: sequences of canonical changes are used to normalise the interpolating Hamiltonian order by order. This order is not the usual one (used for example in [5]) but is more complicated and adapted to the specific problem. The order is defined using a grading function and used to simplify manipulations with formal series in several variables. Unlike [16] our grading functions are mostly non-linear and therefore a sum of terms of a fixed order is neither homogeneous nor quasi-homogeneous.

Before proceeding to the technical statement of our main theorems, we review some results from the classical normal form theory. These results are adapted to the study of area-preserving maps near a fixed point. An introduction to the general theory can be easily found elsewhere, e.g., in [17, 6].

Let $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving map with a fixed point at the origin

$$F_0(0) = 0.$$

Since F_0 is area-preserving $\det DF_0(0) = 1$. Therefore we can denote the two eigenvalues of the Jacobian matrix $DF_0(0)$ by μ and μ^{-1}. These eigenvalues are often called *multipliers* of the fixed point. A fixed point is called

- *hyperbolic* if $\mu \in \mathbb{R}$ and $\mu \neq \pm 1$;
- *elliptic* if $\mu \in \mathbb{C} \setminus \mathbb{R}$ (in this case $|\mu| = 1$);
- *parabolic* if $\mu = \pm 1$.

We also note that the word "parabolic" is often used for a generic fixed point with $\mu = 1$.

1.1. Hyperbolic fixed point

From the viewpoint of the normal form theory this case is exceptional since the map can be transformed into its Birkhoff normal form by an analytic change of variables. Indeed, Moser [18, 19] proved that there is a canonical (*i.e.,* area- and orientation-preserving)

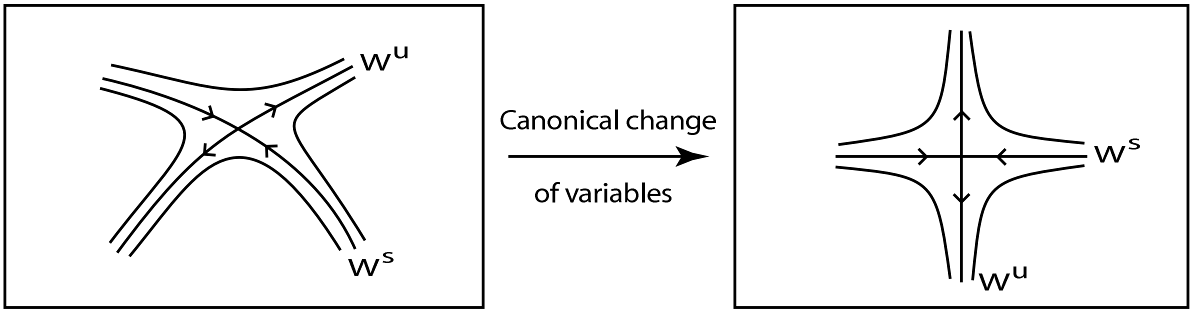

Figure 1. Normal form coordinates in a neighbourhood of a hyperbolic fixed point.

analytical change of variables such that in a neighbourhood of the origin F_0 takes the form $(u, v) \mapsto (u_1, v_1)$ where

$$\begin{cases} u_1 = a(uv)u \\ v_1 = \dfrac{v}{a(uv)} \end{cases}$$

and a is an analytic function of a single variable uv. Moreover, $a(0) = \mu$. It is interesting to note that although the change of the variables is not unique the function a is uniquely defined by the map F_0. The normal form is integrable since $u_1 v_1 = uv$. Therefore a neighbourhood of the fixed point is foliated into invariant lines as shown in Figure 1. Moreover, in a neighbourhood of the fixed point F_0 coincides with the time-one map of an analytic Hamiltonian system with one degree of freedom:

$$F_0 = \Phi_H^1 .$$

In the coordinates (u, v) the Hamiltonian H is a function of the product uv only and is given explicitly by the following integral:

$$H(uv) = \int^{uv} \log a .$$

We note that H is a local integral only and hence does not imply integrability of the map F_0. On the other hand it provides a powerful tool for studying dynamics of area-preserving maps and is very useful in the problems related to separatrices splitting (see e.g. [13, 14]).

1.2. Elliptic fixed point

As the map is real-analytic the second multiplier $\mu^{-1} = \mu^*$, where μ^* is the complex conjugate of μ. Consequently the multipliers of an elliptic fixed point belong to the unit circle $|\mu| = 1$. Note the assumption $\mu \notin \mathbb{R}$ excludes $\mu = \pm 1$. There is a linear area-preserving change of variables such that the Jacobian of F_0 takes the form of a rotation:

$$DF_0(0) = R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{1.1}$$

where the rotation angle α is related to the multiplier: $\mu = e^{i\alpha}$. The following theorem is a classical result of the normal form theory.

Theorem 1.1 *(Birkhoff normal form) There exists a formal canonical change of variables Φ such that the map*

$$N = \Phi \circ F_0 \circ \Phi^{-1}$$

commutes with the rotation R_α:

$$N \circ R_\alpha = R_\alpha \circ N.$$

The map N is called a *Birkhoff normal form* of F_0 (see e.g. [1, 8]). The map $R_{-\alpha} \circ N$ is tangent to identity, *i.e.*, its Taylor series starts with the identity map. A tangent to identity symplectic map can be formally represented as a time-one map of an autonomous Hamiltonian system: there is a formal Hamiltonian H such that

$$N = R_\alpha \circ \Phi_H^1 \,, \tag{1.2}$$

where Φ_H^1 is a formal time-one map. For the sake of completeness we will provide a proof of this statement later. The corresponding vector field is usually called *Takens normal form vector field*. The Hamiltonian inherits the symmetry of the normal form:

$$H \circ R_\alpha = H \,. \tag{1.3}$$

The symmetry of the normal form has an important corollary: H is a formal integral of N:

$$H \circ N = H \circ R_\alpha \circ \Phi_H^1 = H \circ \Phi_H^1 = H \,.$$

Therefore the Birkhoff normal form N is integrable. Coming back to the original variables we obtain a formal integral of the original map.

Usually the series involved in the construction of the normal form do not converge. On the other hand it is possible to construct an analytic change of variables which transforms the map into the normal form up to a reminder of an arbitrarily high order. In other words, for any $p > 0$ there is a canonical analytic change of coordinates $\tilde\Phi_p$ such that its Taylor series coincides with the formal series Φ up to the order p. Then Taylor series of $\tilde N_p = \tilde\Phi_p \circ F_0 \circ \tilde\Phi_p^{-1}$ coincides with the formal series N up to the order p, and the map $\tilde N_p$ is in the normal form up to a reminder of order $p+1$, *i.e.*,

$$\tilde N_p \circ R_\alpha - R_\alpha \circ \tilde N_p = O(r^{p+1})$$

where $r = \sqrt{x^2 + y^2}$. Moreover,

$$\tilde N_p = \Phi_{H_p}^1 + O(r^{p+1})$$

where H_p is a polynomial Hamiltonian obtained by neglecting all terms of orders higher than p in the formal series H.

Alternatively, it is possible to construct a smooth (C^∞) change of variables such that the remainders are flat.

The transformation to the normal form is not unique. Traditionally this freedom is used to eliminate some coefficients from the normal form map N. We will take a slightly different point of view and simplify the series for the formal interpolating Hamiltonian. In this way the problem is reduced to a study of a normal form for a Hamiltonian system with symmetry. We note that this is a classical subject and lower order normal forms can be found in the literature (see for example [15, 4]).

Our goal is to achieve a substantial simplification for all orders and to derive the unique normal form for the generic case. The results depend on the rotation angle α defined in (1.1).

Definition 1.2 *A fixed point is called* resonant *if there exists $n \in \mathbb{N}$ such that $\mu^n = 1$. The least positive n is called the* order of the resonance*. If the fixed point is not resonant we call it* non-resonant*. A resonant fixed point is called* strongly resonant *if $n \leq 4$. Otherwise it is called* weakly resonant*.*

The resonances of orders one and two are related to a parabolic and not elliptic fixed point.

It is convenient to introduce the symplectic polar coordinates (I, φ) by

$$\begin{aligned} x &= \sqrt{2I}\cos\varphi\,, \\ y &= \sqrt{2I}\sin\varphi\,. \end{aligned}$$

If the fixed point is not resonant the normal form is a rotation:

$$N = R_{\omega(I)} \qquad \text{where} \qquad \omega(I) = \alpha + \sum_{k\geq 1} \omega_k I^k\,.$$

Note that the rotation angle depends on the action I. The coefficients ω_k are defined uniquely and provide a full set of formal invariants for F_0. Writing down Hamiltonian equations and comparing their solution with the map $R_{-\alpha} \circ N = R_{\omega(I)-\alpha}$, we can easily check that the formal interpolating Hamiltonian is defined by

$$\frac{\partial H}{\partial I} = \omega(I) - \alpha\,, \qquad \frac{\partial H}{\partial \varphi} = 0\,.$$

Therefore it has the form

$$H(I, \varphi) = \sum_{k\geq 1} \frac{\omega_k}{k+1} I^{k+1} \equiv I^2 A(I)\,.$$

We see that the coefficients of the series $A(I)$ are defined uniquely and can be used to formally classify the maps instead of ω_k.

In the case of a resonant fixed point the leading order of the Hamiltonian H has the form [1]

$$H(I, \varphi) = I^2 A(I) + I^{n/2} B(I) \cos n\varphi + O(I^n)$$

where A and B are polynomial in I. In this paper we will show that this form can be extended up to all orders and study the uniqueness of the coefficients. The following theorem is the main result of the paper.

Theorem 1.3 *If F_0 is a smooth (C^∞ or analytic) area preserving map with a resonant elliptic fixed point of order n at the origin, then there is a formal Hamiltonian H and formal canonical change of variables which conjugates F_0 with $R_\alpha \circ \Phi_H^1$. Moreover, H has the following form:*

- *if $n \geq 4$ and $A(0)B(0) \neq 0$*
$$H(I,\varphi) = I^2 A(I) + I^{n/2} B(I) \cos n\varphi \tag{1.4}$$
where
$$A(I) = \sum_{k\geq 0} a_k I^k \,, \qquad B(I) = \sum_{k\geq 0} b_k I^{2k} \,. \tag{1.5}$$
- *If $n = 3$ and $B(0) \neq 0$*
$$H(I,\varphi) = I^3 A(I) + I^{3/2} B(I) \cos 3\varphi \tag{1.6}$$
where
$$A(I) = \sum_{\substack{k\geq 0\\ k\neq 2 \pmod 3}} a_k I^k \,, \qquad B(I) = \sum_{\substack{k\geq 0\\ k\neq 2 \pmod 3}} b_k I^k \,. \tag{1.7}$$

The coefficients of the series A and B are defined uniquely by the map F_0 provided the leading order is normalised to ensure $b_0 > 0$.

Note that the sign of b_0 changes after a substitution $\varphi \mapsto \varphi + \frac{\pi}{n}$. In the theorem the change of the variables is note unique.

The theorem provides a formal classification for the generic maps with respect to formal canonical changes of variables.

Theorem 1.3 follows from Propositions 4.3, 5.1 and 6.1 which state equivalent results in complex coordinates for $n = 3$, $n = 4$ and $n \geq 5$ respectively.

1.3. Unique normal forms for families

Now instead of an individual area-preserving map F_0 we consider a family of area-preserving maps F_ε. It can be either analytic, C^∞ smooth or formal. We assume that the origin is an elliptic fixed point of F_0. The implicit function theorem implies that there is $\varepsilon_0 > 0$ such that in a neighbourhood of the origin F_ε has an elliptic fixed point for all $|\varepsilon| < \varepsilon_0$. Without loosing in generality we can assume that the fixed point has already been moved to the origin.

Let α_0 be the rotation angle defined by $DF_0(0)$. It is well known that the family can be transformed to the normal form in a way similar to an individual map. For example, one can consider ε as an additional variable and extend the map to a higher dimension by adding the line $\varepsilon \mapsto \varepsilon$. Then the standard Birkhoff normal form arguments can be applied.

In our case there is a formal Hamiltonian χ_ε such that
$$N_\varepsilon = \Phi_{\chi_\varepsilon}^{-1} \circ F_\varepsilon \circ \Phi_{\chi_\varepsilon}^1$$

is in the normal form, *i.e.*, $N_\varepsilon \circ R_{\alpha_0} = R_{\alpha_0} \circ N_\varepsilon$. We see that although the rotation angle of the fixed point may change with ε the normal form keeps the symmetry defined by α_0. The normal form can be formally interpolated by an autonomous Hamiltonian flow:

$$N_\varepsilon = R_{\alpha_0} \circ \Phi^1_{H_\varepsilon}$$

where H_ε is a symmetric formal Hamiltonian

$$H_\varepsilon = H_\varepsilon \circ R_{\alpha_0} .$$

It is well known that these statements can be proved by an appropriate modification of the arguments used for the case of individual maps. The leading order of the normal form has the form [1]

$$H_\varepsilon(I,\varphi) = IA(I,\varepsilon) + I^{n/2}B(I,\varepsilon)\cos n\varphi + O(I^{n/2+1}) ,$$

where n is the order of the resonance, i.e., the smallest natural number such that $\mathrm{e}^{i\alpha_0 n} = 1$. It is also well known the changes of variables involved in the construction of the normal forms are not unique. This freedom can used to provide further simplifications of the normal form Hamiltonian H_ε.

Theorem 1.4 *If F_ε is a smooth (C^∞ or analytic) family of area preserving maps such that F_0 has a resonant elliptic fixed point of order n at the origin, then there is a formal Hamiltonian H_ε and formal canonical change of variables which conjugates F_ε with $R_{\alpha_0} \circ \Phi^1_{H_\varepsilon}$, where α_0 is the rotation angle defined by $DF_0(0)$. Moreover, H_ε has the following form:*

$$H_\varepsilon(I,\varphi) = IA(I,\varepsilon) + I^{n/2}B(I,\varepsilon)\cos n\varphi \tag{1.8}$$

where H_ε for $\varepsilon = 0$ coincides with H of Theorem 1.3 and

- *if $n \geq 4$ and $\partial_I A(0,0) \cdot B(0,0) \neq 0$*

$$A(I,\varepsilon) = \sum_{k,m\geq 0} a_{k,m} I^k \varepsilon^m , \qquad B(I,\varepsilon) = \sum_{k,m\geq 0} b_{k,m} I^{2k} \varepsilon^m , \tag{1.9}$$

  where $a_{0,0} = 0$,

- *if $n = 3$ and $B(0,0) \neq 0$*

$$A(I,\varepsilon) = \sum_{\substack{k,m\geq 0 \\ k\neq 1 \pmod 3}} a_{k,m} I^k \varepsilon^m , \qquad a_{0,0} = a_{1,0} = 0 , \tag{1.10}$$

$$B(I,\varepsilon) = \sum_{\substack{k,m\geq 0 \\ k\neq 2 \pmod 3}} b_{k,m} I^k \varepsilon^m . \tag{1.11}$$

The coefficients of the series A and B are defined uniquely by the map F_ε provided the leading order is normalized to ensure $b_{00} > 0$.

1.4. Structure of the paper

The rest of the paper is structured in the following way. In Section 2 we provide some useful definitions, describe the usage of $(z,\bar{z})$ variables and derive several useful formulae. In Section 3 we prove that a tangent to identity area-preserving map can be formally interpolated by an autonomous Hamiltonian. This result is used in the proof of the main theorem and is included for completeness of the arguments. Finally, in Sections 4, 5 and 6 we derive the unique normal forms for the cases $n \geq 5$, $n = 4$ and $n = 3$ respectively.

Finally, in Section 7 we construct unique normal forms for the families of area-preserving maps.

2. Lie series in the complex form

2.1. Complex variables

It is well known that the normal form theory can look much simpler in the complex coordinates z and $\bar{z}$ defined by

$$z = x + iy \qquad \text{and} \qquad \bar{z} = x - iy. \tag{2.1}$$

The change $(x,y) \mapsto (z,\bar{z})$ is a linear automorphism of $\mathbb{C}^2$ with the inverse transformation given by

$$x = \frac{z+\bar{z}}{2} \qquad \text{and} \qquad y = \frac{z-\bar{z}}{2i}\,. \tag{2.2}$$

If x and y are both real then $\bar{z}$ is the complex conjugate of z, i.e., $\bar{z} = z^*$.

It is important to note that the change is not symplectic since

$$dx \wedge dy = -\frac{1}{2i} dz \wedge d\bar{z}\,. \tag{2.3}$$

Nevertheless since the Jacobian is constant, an area-preserving map also preserves the form $dz \wedge d\bar{z}$.

Let $(f,\bar{f})$ be the components of a map F in the coordinates $(z,\bar{z})$. We say that F has *real symmetry* if

$$\bar{f}(z,\bar{z}) = \left(f(\bar{z}^*, z^*)\right)^*. \tag{2.4}$$

Then the second component of F can be restored using this symmetry and we do not have to consider it separately.

We note that F commutes with the involution $(z,\bar{z}) \mapsto (\bar{z}^*, z^*)$. In the original coordinates this involution takes the form $(x,y) \mapsto (x^*, y^*)$ and, consequently, F takes real values when both x and y are real.

For a real-analytic map F the real symmetry described by (2.4) can be easily restated in terms of its Taylor coefficients. Then the real symmetry naturally extends from real-analytic functions onto formal power series and suggests the following definition.

Definition 2.1 *We say that the vector-valued series* $(f(z,\bar z),\bar f(z,\bar z))$ *where* f *and* $\bar f$ *are formal series of the form*

$$f(z,\bar z)=\sum_{k,l\ge 0} f_{kl}z^k\bar z^l \qquad \textit{and} \qquad \bar f(z,\bar z)=\sum_{k,l\ge 0}\bar f_{kl}z^k\bar z^l\,,$$

has the real symmetry *if*

$$\bar f_{kl}=f^*_{lk} \tag{2.5}$$

for all $k,l\ge 0$*.*

This trick substantially simplifies manipulations with power series. As an example consider the rotation $(x,y)\mapsto R_\alpha(x,y)$ defined by equation (1.1). In the complex notation this map takes the diagonal form $(z,\bar z)\mapsto(\mu z,\mu^*\bar z)$. Indeed, for the first component we get

$$\begin{aligned} z=x+iy \;\mapsto\; z_1 &= (x\cos\alpha-y\sin\alpha)+i\,(x\sin\alpha+y\cos\alpha)\\ &= (x+iy)\,(\cos\alpha+i\sin\alpha)=e^{i\alpha}z=\mu z\,. \end{aligned}$$

The formula for the $\bar z$ component of the map is obtained by the real symmetry.

We will also consider scalar functions and scalar formal series rewritten in terms of the variables $(z,\bar z)$. If h is real-analytic in a neighbourhood of the origin, it can be expanded in Taylor series

$$h(z,\bar z)=\sum_{k,l\ge 0}h_{kl}z^k\bar z^l\,.$$

Since $h(z,z^*)$ is real for all $z\in\mathbb{C}$ such that the series converges, the coefficients are symmetric:

$$h_{kl}=h^*_{lk}$$

for all k,l. This prompts the following definition.

Definition 2.2 *We say that a formal power series*

$$h=\sum_{k,l\ge 0}h_{kl}z^k\bar z^l$$

is real-valued if $h_{kl}=h^*_{lk}$ *for all* k,l*.*

2.2. Divergence-free vector fields in the complex form

From now on we assume that the $\bar f$ component is obtained from f using the real symmetry. It is convenient to introduce the divergence operator by

$$\operatorname{div} f=\frac{\partial f}{\partial z}+\frac{\partial \bar f}{\partial \bar z}\,. \tag{2.6}$$

In the future proofs we will need the following simple fact.

Lemma 2.3 *A map $(z,\bar z)\mapsto (f(z,\bar z),\bar f(z,\bar z))$ with real symmetry is area-preserving if and only if the function $g(z,\bar z):=f(z,\bar z)-z$ satisfies*

$$\operatorname{div} g=\{\bar g,g\} \tag{2.7}$$

where $\bar g$ is obtained from g using the real symmetry.

Proof. Taking into account $f(z,\bar z)=z+g(z,\bar z)$ and using the real symmetry to get $\bar f$ we obtain

$$\begin{aligned} df\wedge d\bar f &= \left(dz+\frac{\partial g}{\partial z}dz+\frac{\partial g}{\partial \bar z}d\bar z\right)\wedge\left(d\bar z+\frac{\partial \bar g}{\partial \bar z}d\bar z+\frac{\partial \bar g}{\partial z}dz\right)\\ &=\left(1+\frac{\partial \bar g}{\partial \bar z}+\frac{\partial g}{\partial z}+\frac{\partial g}{\partial z}\frac{\partial \bar g}{\partial \bar z}-\frac{\partial g}{\partial \bar z}\frac{\partial \bar g}{\partial z}\right)dz\wedge d\bar z. \end{aligned}$$

Since f is area-preserving $df\wedge d\bar f=dz\wedge d\bar z$ and we get the identity

$$\frac{\partial g}{\partial z}+\frac{\partial \bar g}{\partial \bar z}=\frac{\partial g}{\partial \bar z}\frac{\partial \bar g}{\partial z}-\frac{\partial g}{\partial z}\frac{\partial \bar g}{\partial \bar z}$$

which is equivalent to (2.7). □

Let us consider the Hamiltonian equations

$$\begin{aligned}\dot z&=-2i\frac{\partial h}{\partial \bar z},\\ \dot{\bar z}&=2i\frac{\partial h}{\partial z}.\end{aligned}$$

Obviously this vector field has zero divergence. Let us consider a vector field

$$\dot z=g(z,\bar z)\qquad \dot{\bar z}=\bar g(z,\bar z)\,,$$

where $\bar g$ is obtained from g using the real symmetry. A natural question arises: Suppose g is divergence free, is it Hamiltonian with a real-valued Hamiltonian function?

The next Lemma gives a positive answer for polynomial (and consequently for all formal) vector fields.

Lemma 2.4 *Let g_p be a homogeneous polynomial of order $p\ge 0$. There is a real-valued homogeneous polynomial h_{p+1} of order $p+1$ such that*

$$g_p=-2i\frac{\partial h_{p+1}}{\partial \bar z}, \tag{2.8}$$

if and only if

$$\operatorname{div} g_p=0\,. \tag{2.9}$$

If exists, the polynomial h_{p+1} is unique in the class of real-valued homogeneous polynomials of z and $\bar z$. Moreover, if for some μ with $|\mu|=1$ we have $g_p(\mu z,\mu^*\bar z)=\mu g_p(z,\bar z)$ then $h_{p+1}(\mu z,\mu^*\bar z)=h_{p+1}(z,\bar z)$.

Proof. Suppose

$$g_p(z,\bar z)=\sum_{k+l=p}a_{kl}z^k\bar z^l$$

is divergence free. Then

$$0 = \operatorname{div} g_p = \frac{\partial g_p}{\partial z} + \frac{\partial \bar g_p}{\partial \bar z} = \sum_{k+l=p} k a_{kl} z^{k-1} \bar z^l + l a_{lk}^* z^k \bar z^{l-1}$$

where we used the real symmetry to get $\bar g_p$. Collecting the coefficients in front of $z^k \bar z^l$ we see that $\operatorname{div} g_p = 0$ if and only if

$$(k+1) a_{k+1,l} + (l+1) a_{l+1,k}^* = 0 \tag{2.10}$$

for all $k, l \geq 0$. These relations involve all coefficients excepting a_{0p}. A homogeneous polynomial of order $p+1$ has the form

$$h_{p+1} = \sum_{k+l=p+1} h_{kl} z^k \bar z^l \,. \tag{2.11}$$

Substituting this sum into (2.8), we easily see that h_{p+1} satisfies the equation if and only if

$$h_{kl} = -\frac{a_{k,l-1}}{2il}$$

for all $l \geq 1$. These equalities define all coefficients of h_{p+1} excepting $h_{p+1,0}$. Equation (2.10) implies the real-valuedness conditions

$$h_{kl} = h_{lk}^* \tag{2.12}$$

for $l \geq 1$. The coefficient $h_{p+1,0}$ is not defined by the equation so we set $h_{p+1,0} = h_{0,p+1}^*$ to extend Equation (2.12) onto $l = 0$. Then h_{p+1} is real valued.

The other direction of the lemma is trivial since a Hamiltonian vector field is divergence free.

Finally, if g commutes with the rotation $z \mapsto \mu z$ the Hamiltonian h_{p+1} is invariant with respect to this rotation due to the explicit formula for its coefficients provided above. □

2.3. Formal Lie series

Let χ and g be two formal power series. We note that any of the series involved in next definitions may diverge. The linear operator defined by the formula

$$L_\chi g = -2i \left\{ g, \chi \right\}_{z,\bar z} \tag{2.13}$$

is called *the Lie derivative* generated by χ. We note that if χ starts with order p and g starts with order q, then the series $L_\chi g$ starts with order $p+q-2$ as the Poisson bracket involves differentiation. We assume $p \geq 3$. Then the lowest order in $L_\chi g$ is at least $q+1$ and we can define the exponent of L_χ by

$$\exp(L_\chi) g = g + \sum_{k\geq 1} \frac{1}{k!} L_\chi^k g \,, \tag{2.14}$$

where L_χ^k stands for the operator L_χ applied k times. The lowest order in the series $L_\chi^k g$ is at least $q+k$ and consequently every coefficient of $\exp(L_\chi) g$ depends only on a finite number of coefficients of χ and g. More precisely, a coefficient of order n depends polynomially on the coefficients of orders up to n.

Let $\mathrm{id} : \mathbb{C}^2 \to \mathbb{C}^2$ be the identity map and consider the formal series

$$\Phi^1_\chi = \exp(L_\chi)\mathrm{id}$$

where the exponential is applied componentwise. We note that it is easy to construct the formal series for the inverse map:

$$\Phi^{-1}_\chi = \exp(-L_\chi)\mathrm{id}\,.$$

It follows from the following more general relation: for any formal series g

$$g \circ \Phi^1_\chi = \exp(L_\chi) g\,. \tag{2.15}$$

Indeed, this equality is known to be valid for convergent series. In particular, for a polynomial χ the series Φ^1_χ converges in a neighbourhood of the origin as it coincides with the Taylor expansion of the time-one map which shifts a point along trajectories of the Hamiltonian equations:

$$\begin{aligned} \dot z &= -2i\frac{\partial \chi}{\partial \bar z},\\ \dot{\bar z} &= 2i\frac{\partial \chi}{\partial z}. \end{aligned}$$

The factor $2i$ appears due to the symplectic form (2.3).

We can substitute a solution of the Hamiltonian equation into a function $g(z,\bar z)$. The Hamiltonian equations and the chain rule imply

$$\dot g = 2i\,\{\chi, g\}\,.$$

Then repeating the arguments inductively we get a formula for a derivative of order n in t:

$$g^{(n)} = L^n_\chi g.$$

Writing a Taylor series centred at 0 for $g \circ \Phi^t_\chi$ and substituting $t = 1$ we obtain (2.15). Then the formula extends from polynomials onto formal series since each order of the series depends only on a finite number of coefficients.

3. Formal interpolation

The next theorem states that a tangent to identity area-preserving map can be formally interpolated by a Hamiltonian flow. Note that the theorem says nothing about convergence of the series, even in the case when the original map is analytic.

Theorem 3.1 *If a formal series*

$$f(z,\bar z) = z + \sum_{\substack{k+l\ge 2\\ k,l\ge 0}} c_{kl} z^k \bar z^l$$

is a first component of an area-preserving map with the real symmetry then there exists a unique real-valued formal Hamiltonian

$$h(z,\bar z)=\sum_{\substack{k+l\ge 3\\ k,l\ge 0}} h_{kl} z^k \bar z^l$$

such that

$$f=\exp(L_h)z.$$

Proof. First we introduce the notation. Let h_k, $k\ge 3$, be a homogeneous polynomial of order k and for $m\ge 3$

$$H_m=\sum_{k=3}^{m} h_k(z,\bar z)\,.$$

Let $[\,\cdot\,]_k$ denote terms of order k in a formal series. For example, $[H_m]_k = h_k$ for $3\le k\le m$ and $[H_m]_k=0$ for $k>m$. Consider the series

$$\phi^1_{H_m}=\exp(L_{H_m})z\,,$$

where z stands for the function $(z,\bar z)\mapsto z$. We will need a more explicit formula for the exponential map. It is convenient to introduce

$$L_s g := 2i\,\{h_{s+2},g\}$$

for the Lie derivative generated by the homogeneous polynomial h_{s+2}. If g is a homogeneous polynomial of order q, then $L_s(g)$ is a homogeneous polynomial of order $q+s$. Therefore L_s increases the order of any homogeneous polynomial by s. Then using the bi-linearity of the Poisson bracket we get

$$L_{H_m} g = 2i\,\{H_m,g\} = 2i\sum_{k=3}^{m}\{h_k,g\} = \sum_{s=1}^{m-2} L_s g\,.$$

Substituting this sum into the series for the exponential map we obtain

$$\begin{aligned}\phi^1_{H_m} &= z + L_{H_m} z + \sum_{l\ge 2}\frac{1}{l!}L^l_{H_m} z\\ &= z+\sum_{s=1}^{m-2} L_s z + \sum_{l\ge 2}\frac{1}{l!}\sum_{1\le s_1,\dots,s_l\le m-2} L_{s_1}\cdots L_{s_l} z.\end{aligned}$$

Now for every $k\ge 1$ we collect the terms of order $k+1$

$$\left[\phi^1_{H_m}\right]_{k+1} = L_k z + \sum_{l=2}^{k}\frac{1}{l!}\sum_{\substack{s_1+\ldots+s_l=k\\ 1\le s_1,\dots,s_l\le m-2}} L_{s_1}\ldots L_{s_l} z. \tag{3.1}$$

We need one more auxiliary formula. Let us write $f_p=[f]_p$. We get

$$f(z,\bar z) = z+\sum_{p\ge 2} f_p(z,\bar z).$$

Substituting this series into equation (2.7) of Lemma 2.3 and collecting terms of order $p-1$ we get

$$\operatorname{div} f_p = \sum_{k=2}^{p-1} \left\{ \bar f_k, f_{p-k+1} \right\} . \tag{3.2}$$

Our aim is to construct an infinite sequence of h_k such that for every $m \geq 3$

$$\left[\phi^1_{H_m}\right]_k = [f]_k \qquad \text{for } k = 2, \ldots, m-1. \tag{3.3}$$

We use the induction. First consider $m = 3$. Then equation (3.3) reads

$$L_1 z = f_2$$

which is equivalent to

$$-2i \frac{\partial h_3}{\partial \bar z} = f_2. \tag{3.4}$$

We note that the sum in the right-hand side of equation (3.2) with $p = 2$ has no terms and consequently $\operatorname{div} f_2 = 0$. Then Lemma 2.4 implies that there exists a unique real-valued h_3 which satisfies equation (3.4). This choice of h_3 guaranties that (3.3) is satisfied for $m = 3$.

We start the induction step. Suppose for some $m \geq 3$ we have found $h_3, \ldots, h_m$ such that (3.3) holds. Now we look for h_{m+1} such that (3.3) holds with m replaced by $m+1$. Equation (3.1) and the induction assumption imply that

$$[\phi^1_{H_{m+1}}]_k = [\phi^1_{H_m}]_k = f_k \qquad \text{for } k \leq m-1.$$

Then equation (3.1) implies that the equality $[\phi^1_{H_{m+1}}]_m = f_m$ is equivalent to

$$-2i \frac{\partial h_{m+1}}{\partial \bar z} + \sum_{l=2}^{m-1} \frac{1}{l!} \sum_{\substack{s_1+\ldots+s_l=m-1 \\ 1\leq s_1,\ldots,s_l \leq m-2}} L_{s_1} \ldots L_{s_l} z = f_m. \tag{3.5}$$

We note that this formula includes $L_s = L_{h_{s+2}}$ with $s \leq m-2$ which depend on h_k with $k \leq m$. Therefore we consider (3.5) as an equation for h_{m+1}. In order to show that the equation has a real-valued solution we need to check the assumptions of Lemma 2.4 are satisfied. Since $\operatorname{div}\left(2i\frac{\partial h_{m+1}}{\partial \bar z}\right) = 0$ it is sufficient to check that $\operatorname{div}[\phi^1_{H_{m+1}}]_m = \operatorname{div} f_m$. The last property follows from the area preservation. Indeed consider two area-preserving maps

$$f(z, \bar z) = z + \sum_{k \geq 2} f_k \qquad \text{and} \qquad \widetilde f(z, \bar z) = z + \sum_{k\geq 2} \widetilde f_k$$

such that $f_j = \widetilde f_j$ for $2 \leq j \leq m-1$. Then (3.2) implies

$$\operatorname{div} f_m = \sum_{j=2}^{m-1} \left\{ \bar f_j, f_{k-j} \right\} = \sum_{j=2}^{m-1} \left\{ \bar{\tilde f}_j, \tilde f_{k-j} \right\} = \operatorname{div} \tilde f_m.$$

We apply this argument with $\tilde f = \phi^1_{H_{m+1}}$ which is area-preserving by Liouville's theorem. Therefore equation (3.5) can be uniquely solved with respect to h_{m+1}. The induction step is complete and we have uniquely defined the desired formal Hamiltonian $h = \sum_{p\geq 3} h_p$. □

Remark 3.2 *If the map $f(z,\bar z)=\mu z+\sum_{k\ge2} f_k(z,\bar z)$ is in Birkhoff normal form then its coefficients commute with the rotation* $(z,\bar z)\mapsto(\mu z,\mu^*\bar z)$, i.e., $f_k(\mu z,\mu^*\bar z)=\mu f_k(z,\bar z)$. *The map*

$$\mu^* f(z,\bar z)=z+\sum_{k\ge2}\mu^* f_k(z,\bar z)$$

satisfies the assumptions of the formal interpolation theorem which implies that there exists a unique formal Hamiltonian vector field such that $f=\mu\phi_h^1$. Moreover a more accurate analysis of the proof shows that the Hamiltonian h is in Birkhoff normal form itself. In other words, the formal Hamiltonian is invariant with respect to the rotation:

$$h(\mu z,\mu^*\bar z)=h(z,\bar z).$$

We remind that the Hamiltonian h is a formal series, and this relation is to be interpreted termwise.

Remark 3.3 *We also proved that the map $F=(f,\bar f)$ can be approximated by an integrable map. Indeed, expanding $F-\Phi^1_{H_m}$ into Taylor series we use equation* (3.3) *to show that the first $m-1$ orders of the series vanish. Then the standard estimate for a remainder of the Taylor formula implies that for any $m\ge3$*

$$F=\Phi^1_{H_m}+O(r^m)$$

where $r=|z|+|\bar z|$.

4. Weak resonances

Suppose that an area-preserving map f is in the resonant normal form, *i.e.*, $f(\mu z,\mu^*\bar z)=\mu f(z,\bar z)$. Then there is a formal Hamiltonian such that

$$f=\mu\phi_h^1=\mu\exp(L_h)z$$

and h has the symmetry induced by the linear part of f:

$$h(\mu z,\mu^*\bar z)=h(z,\bar z)\,.$$

The symmetry of the interpolating Hamiltonian h implies that it is represented by a formal series which contains resonant terms only:

$$h(z,\bar z)=\sum_{\substack{k+l\ge3\\ k=l \pmod n}} h_{kl}z^k\bar z^l.$$

It is easy to see that these series involve a fourth order term $h_{22}z^2\bar z^2$ independently of n. If $n\ge5$ there are no other resonant term of an order 4 or less. Then the leading order is of the same form as in the non-resonant case. For this reason the resonances with $n\ge5$ are called *weak*.

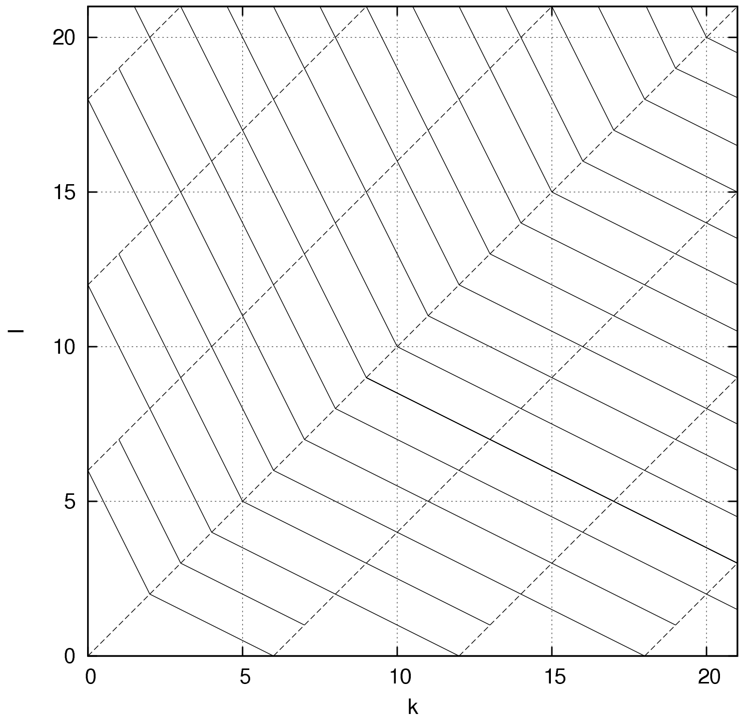

Figure 2. Resonant terms for $n = 6$. On the diagram the resonant terms correspond to intersections of solid and dashed lines on the (k, l) plane. Each of the solid lines connects the points of equal δ-order.

Let us consider the case $n \geq 5$ with more details:

$$h(z, \bar{z}) = h_{22} z^2 \bar{z}^2 + \sum_{\substack{k+l \geq 5 \\ k=l \pmod{n}}} h_{kl} z^k \bar{z}^l. \tag{4.1}$$

We will simplify these series using canonical substitutions.

It is convenient to group together terms of the same δ-order. For a monomial we define its δ-order by

$$\delta(z^k \bar{z}^l) = 2 \left| \frac{k-l}{n} \right| + \min\{ k, l \} = \frac{1}{2}(k+l) - \frac{n-4}{2n} |k-l| . \tag{4.2}$$

Let $\mathcal{H}_m^n$ denote the set of all real-valued polynomials which can be represented as a sum of resonant monomials of the δ-order m. For example, $\mathcal{H}_2^n$ consists of polynomials of the form $c_{n0} z^n + c_{22} z^2 \bar{z}^2 + c_{0n} \bar{z}^n$ with $c_{0n} = c_{n0}^* \in \mathbb{C}$ and $c_{22} \in \mathbb{R}$. Therefore $\mathcal{H}_2^n$ is a three dimensional real vector space. The resonant terms are sketched on Figure 2.

Let $\lfloor \cdot \rfloor$ denote the integer part of a number. Assume $m \geq 1$.

Lemma 4.1 *The set* $\mathcal{H}_m^n$ *is a real vector space of dimension* $1 + 2\left\lfloor \frac{m}{2} \right\rfloor$.

Proof. If a resonant monomial $z^k \bar{z}^l$ has the δ-order m then

$$\begin{aligned} k \ &= l + nj \qquad \text{for some } j \in \mathbb{Z} \text{ (resonant term)}, \\ m &= 2|j| + \min\{ k, l \} \qquad (\delta\text{-order equals } m). \end{aligned}$$

Let us count the number of monomials which satisfy these two conditions. There is one with $j=0$: $k=l=m$. Then there are $\lfloor \frac{m}{2} \rfloor$ monomials with $j>0$. Indeed, since $k>l$ we get

$$\begin{aligned} l &= m-2j \qquad 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor , \\ k &= l+nj\,. \end{aligned}$$

We also get the equal number of monomials with $j<0$ due to the symmetry. It is convenient to denote the resonant monomials by

$$Q_{m,j} = z^{m+nj-2j}\bar z^{m-2j} \qquad \text{and} \qquad Q_{m,-j} = z^{m-2j}\bar z^{m+nj-2j} \tag{4.3}$$

for $0 \le j \le \lfloor \frac{m}{2} \rfloor$. Then any resonant polynomial which contains only monomials of the δ-order m has the form $\sum_{j=-\lfloor \frac{m}{2} \rfloor}^{\lfloor \frac{m}{2} \rfloor} c_j Q_{mj}$. Taking into account that $c_{kl} = c^*_{lk}$ due to real-valuedness, we conclude that the real dimension of the space $\dim \mathcal{H}^n_m = 1 + 2\lfloor \frac{m}{2} \rfloor$. □

Lemma 4.2 *Let $n \ge 4$. If $z^{k_1}\bar z^{l_1}$ and $z^{k_2}\bar z^{l_2}$ are two resonant monomials of δ-orders m_1 and m_2 respectively, then*

$$\{z^{k_1}\bar z^{l_1}, z^{k_2}\bar z^{l_2}\} = (k_1 l_2 - k_2 l_1) z^{k_1+k_2-1}\bar z^{l_1+l_2-1}$$

is a resonant monomial of δ-order $m \ge m_1 + m_2 - 1$.

Proof. The Poisson bracket of the monomials has the form $\mathrm{const}\, z^k \bar z^l$ with $k = k_1 + k_2 - 1$ and $l = l_1 + l_2 - 1$. Using the second formula from the definition of the δ order (4.2) we get

$$\begin{aligned} m &= \frac{1}{2}(k+l) - \frac{n-4}{2n}|k-l| \\ &= \frac{1}{2}(k_1+k_2+l_1+l_2-2) - \frac{n-4}{2n}|k_1+k_2-l_1-l_2|\,. \end{aligned}$$

Then we rewrite it in the form

$$\begin{aligned} m = \frac{1}{2}(k_1+l_1) - \frac{n-4}{2n}|k_1-l_1| + \frac{1}{2}(k_2+l_2) - \frac{n-4}{2n}|k_2-l_2| - 1 \\ + \frac{n-4}{2n}\left(|k_1-l_1| + |k_2-l_2| - |k_1-l_1+k_2-l_2|\right)\,. \end{aligned}$$

Since the last parenthesis is not negative and $n \ge 4$ we conclude

$$\begin{aligned} m &\ge \frac{1}{2}(k_1+l_1) - \frac{n-4}{2n}|k_1-l_1| + \frac{1}{2}(k_2+l_2) - \frac{n-4}{2n}|k_2-l_2| - 1 \\ &= m_1 + m_2 - 1 \end{aligned}$$

which completes the proof of the lemma. □

Obviously a similar statement is valid for a product of two monomials.

Now we state a proposition which is the central part of our main theorem.

Proposition 4.3 *If $n \ge 5$ and $h_{22}, h_{n0} \ne 0$ there exists a formal canonical change of variables which transforms a formal real-valued Hamiltonian (4.1) into*

$$\tilde h := (z\bar z)^2 A(z\bar z) + (z^n + \bar z^n) B(z^2 \bar z^2), \tag{4.4}$$

where $A, B \in \mathbb{R}[[z\bar{z}]]$ (formal series with real coefficients in the single variable $z\bar{z}$) and $A(0) = h_{22}$, $B(0) = |h_{n0}|$. Moreover, the coefficients of the series A and B are defined uniquely.

Proof. The proposition is proved by induction. We perform a sequence of canonical coordinate changes normalising one δ-order of the formal Hamiltonian at a time.

Let us write $[h]_p$ to denote the terms of the δ-order p in the formal series h. In particular, $[h]_2 = h_{n0}z^n + h_{22}z^2\bar{z}^2 + h_{0n}\bar{z}^n$. The rotation $z \mapsto z\exp(-i\arg(h_{n0})/n)$ transforms it into

$$h_2 := [h]_2 = b_0 z^n + a_0 z^2\bar{z}^2 + b_0\bar{z}^n \tag{4.5}$$

where $b_0 = |h_{n0}|$ and $a_0 = h_{22}$ are both real and positive. We keep the same letter h for the transformed Hamiltonian hoping that it will cause no confusion. After the substitution the leading term of h has the desired form.

All other substitutions are constructed using Lie series (see e.g. [11]). Take a polynomial $\chi_p \in \mathcal{H}_p^n$, $p \geq 2$, and make a substitution generated by χ_p. By (2.15), the new Hamiltonian takes the form

$$\tilde{h} = h + L_{\chi_p}h + \sum_{k\geq 2}\frac{1}{k!}L_{\chi_p}^k h\,.$$

Lemma 4.2 implies that the series $L_{\chi_p}^k h$ starts with the δ-order $k(p-1)+2$ or higher. Therefore each term of $\tilde{h}$ depends only on a finite number of terms in the series h. Moreover, for $2 \leq m \leq p$ we get

$$[\tilde{h}]_m = [h]_m$$

and

$$[\tilde{h}]_{p+1} = [h]_{p+1} + \left[L_{\chi_p}h_2\right]_{p+1}\,. \tag{4.6}$$

We choose χ_p to transform this δ-order to the desired form. For this purpose let us consider the linear operator $L_p : \mathcal{H}_p^n \to \mathcal{H}_{p+1}^n$ defined by

$$L_p\chi_p = \left[L_{\chi_p}h_2\right]_{p+1}\,.$$

It is sometimes called the *homological operator*. We will find a subspace complement to $L_p(\mathcal{H}_p^n)$ in $\mathcal{H}_{p+1}^n$ and choose χ_p to ensure that $[\tilde{h}]_{p+1}$ belongs to this subspace. The properties of L_p are slightly different for odd and even values of p. Let us state these properties first.

If $p = 2k+1$ is odd the kernel of L_p is trivial. Lemma 4.1 implies that $\dim\mathcal{H}_{p+1}^n = \dim\mathcal{H}_p^n + 2$. Therefore

$$\text{co-dim}\,\mathrm{Image}(L_{2k+1}) = 2\,.$$

If $p = 2k$ is even, then $\dim\mathcal{H}_{p+1}^n = \dim\mathcal{H}_p^n$. The kernel of L_p is one-dimensional: it is generated by multiples of h_2^k. Therefore

$$\text{co-dim}\,\mathrm{Image}(L_{2k}) = 1\,.$$

In order to prove these claims and provide an explicit description for the complements, we find a matrix which describes L_p. We note that any polynomial from $\mathcal{H}_p^n$ can be written in the form

$$\chi_p = c_0 Q_{p,0} + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} (c_j Q_{p,j} + c_j^* Q_{p,-j})$$

where c_0 is real and c_j with $j \geq 1$ may be complex. The monomials $Q_{p,j}$ are defined by (4.3). Using (4.5) we compute the action of L_p on monomials:

$$L_p(z^p \bar{z}^p) = -2ib_0 np \left(z^{n+p-1} \bar{z}^{p-1} - z^{p-1} \bar{z}^{n+p-1} \right)$$

and

$$L_p(z^{p-2j+nj} \bar{z}^{p-2j}) = 4ia_0 nj \, z^{p-2j+nj+1} \bar{z}^{p-2j+1} - 2ib_0 n(p-2j) \, z^{p-2j+nj+n-1} \bar{z}^{p-2j-1} \, .$$

These formulae can be rewritten:

$$L_p(Q_{p,0}) = -2ib_0 np Q_{p+1,1} + 2ib_0 np Q_{p+1,-1} \, ,$$
$$L_p(Q_{p,j}) = 4ia_0 nj \, Q_{p+1,j} - 2ib_0 n(p-2j) \, Q_{p+1,j+1} \, , \qquad 1 \leq j \leq \left\lfloor \frac{p}{2} \right\rfloor .$$

Since $L_p \chi_p \in \mathcal{H}_{p+1}^n$ we can represent it in the form

$$L_p \chi_p = d_0 Q_{p+1,0} + \sum_{j=1}^{\lfloor \frac{p+1}{2} \rfloor} (d_j Q_{p+1,j} + d_j^* Q_{p+1,-j})$$

for some constant d_j, $0 \leq j \leq \lfloor \frac{p+1}{2} \rfloor$. From the explicit formulae we see that the image of L_p does not contain terms proportional to $Q_{p+1,0} = z^{p+1} \bar{z}^{p+1}$. Therefore the complement to the image is at least one dimensional. In the image we get $d_0 = 0$ and

$$d_j = 4ia_0 nj \, c_j - 2ib_0 n(p-2j+2) \, c_{j-1} \qquad \text{for} \quad 1 \leq j \leq \left\lfloor \frac{p}{2} \right\rfloor . \tag{4.7}$$

If $p = 2k+1$ is odd, there is an additional equality:

$$d_{k+1} = -2ib_0 n \, c_k \, .$$

In this case the map L_p is considered as an operator which maps $(c_0, \ldots, c_k) \mapsto (d_1, \ldots, d_{k+1})$ and is a linear isomorphism of $\mathbb{C}^{k+1}$. Indeed, the corresponding matrix is triangle and its determinant equals to the product of the diagonal elements: $(-2ib_0 n)^{k+1} (2k)!!$. In this representation the space $\mathcal{H}_p^n$ of real-valued Hamiltonians is identified with the subspace $\{ \operatorname{Im} c_0 = 0 \}$ of $\mathbb{C}^{k+1}$. The operator L_p maps the vector $(i, 0, \ldots, 0)$ into $(2b_0 np, 0, \ldots, 0)$. We see that the preimage of the real-valued polynomial $Q_{p+1,1} + Q_{p+1,-1}$ is not real-valued. Therefore the complement to $L_{2k+1}(\mathcal{H}_{2k+1}^n) \subset \mathcal{H}_{2k+2}^n$ is two dimensional and consists of polynomials of the form

$$d_0 Q_{p+1,0} + d_1 (Q_{p+1,1} + Q_{p+1,-1}) \tag{4.8}$$

with $d_0, d_1 \in \mathbb{R}$.

Now consider the case of $p = 2k$. First we restrict the operator L_{2k} onto the vectors with $c_0 = 0$ and note that $L_{2k} : (0, c_1, \ldots, c_k) \mapsto (0, d_1, \ldots, d_k)$. Equation (4.7) implies that this map is a linear isomorphism (of $\mathbb{C}^k$). Indeed, the corresponding matrix is triangle and its determinant is the product of its diagonal elements: $(4ia_0n)^k\,k! \neq 0$ and the matrix is invertible. Therefore the complement to $L_{2k}(\mathcal{H}^n_{2k}) \subset \mathcal{H}^n_{2k+1}$ is one dimensional and consists of monomials of the form

$$d_0Q_{p+1,0} \tag{4.9}$$

where $d_0 \in \mathbb{R}$ due to real-valuedness.

We conclude that in the homological equation (4.6) the auxiliary polynomial χ_p can be chosen in such a way that $[\tilde{h}]_{p+1}$ is either of the form (4.8) or (4.9).

We continue inductively starting with the δ-order 3. We note that the substitution $\Phi^1_{\chi_p}$ does not change δ-orders $k \le p$ and the composition of the changes is a well-defined formal series. Therefore the original Hamiltonian h can be transformed in such a way that each order is either of the form (4.8) or (4.9). Taking into account the definition of $Q_{p,j}$ we see that h is transformed to the desired form (4.4).

In order to complete the proof we need to establish uniqueness of the series (4.4). We note that the transformation constructed in the first part of the proof is not unique because the kernel of L_{2k} is not empty. Nevertheless the normalised Hamiltonian is unique. Indeed, suppose that two Hamiltonians of the form (4.4) are conjugate, *i.e.*, there is a formal Hamiltonian χ such that

$$\tilde{h}' = \exp(L_\chi)\tilde{h}\,. \tag{4.10}$$

Let p be the lowest δ-order of the formal series χ. Then

$$[\tilde{h}']_m = [\tilde{h}]_m$$

for $2 \le m \le p$ and

$$[\tilde{h}']_{p+1} = [\tilde{h}]_{p+1} + L_p([\chi]_p)\,.$$

Since both $[\tilde{h}']_{p+1}$ and $[\tilde{h}]_{p+1}$ are in the complement to the image of L_p we conclude that $[\tilde{h}']_{p+1} = [\tilde{h}]_{p+1}$ and $L_p([\chi]_p) = 0$. Therefore $[\chi]_p$ is in the kernel of L_p. Since for all odd p the kernel is trivial, p is even. For an even p the kernel is one dimensional and consequently $[\chi]_p = c[h_2^{p/2}]_p$ for some $c \neq 0$. Then obviously $\tilde{h} = \exp(-cL_{\tilde{h}^{p/2}})\tilde{h}$ and we obtain

$$\tilde{h}' = \exp(L_\chi)\exp(-cL_{\tilde{h}^{p/2}})\tilde{h}\,.$$

The composition of two tangent to identity maps is also tangent to identity, and Theorem 3.1 implies that there is a formal series $\tilde{\chi}$ such that

$$\exp(L_{\tilde{\chi}}) = \exp(L_\chi)\exp(-cL_{\tilde{h}^{p/2}})\,.$$

Then $\tilde{h}' = \exp(L_{\tilde{\chi}})\tilde{h}$. We obtained an equation of the form (4.10) but the lowest δ-order of $\tilde{\chi}$ is at least $p+2$. Then the argument can be repeated starting with (4.10) to show that $\tilde{h}$ and $\tilde{h}'$ coincide at all orders. □

5. Forth order resonance

In the case $n = 4$ the construction of the unique normal form is similar to the construction used for the case of a weak resonance. Nevertheless this case is to be considered separately since the matrix of the homological operator is not triangle.

The Hamiltonian h is given by

$$h(z,\bar z) = \sum_{\substack{k+l\geq 4\\ k=l \pmod 4}} h_{kl} z^k \bar z^l .$$

In the case of $n = 4$, the δ-order defined by (4.2) is just a half of the usual order of a polynomial.

The leading terms of the series are of order 4 and correspond to (k, l) equal to $(4, 0)$, $(2, 2)$ and $(0, 4)$. The coefficient h_{22} is real due to the real valuedness. Without loosing in generality we can assume that h_{40} is real and positive which can be achieved by rotating the complex plane using the substitution: $z \mapsto e^{-i\arg(h_{40})/4} z$. Then taking into account the real valuedness of h we can write the terms of order four in the following form:

$$h_2(z,\bar z) = a_0 z^2 \bar z^2 + b_0 (z^4 + \bar z^4), \tag{5.1}$$

where $a_0 = h_{22}$ and $b_0 = |h_{40}|$.

Proposition 5.1 *If $h_{40} \neq 0$, there is a formal canonical change of variables which transforms $h(z, \bar z)$ into*

$$\widetilde{h}(z,\bar z) = z^2\bar z^2 A(z\bar z) + (z^4 + \bar z^4) B(z^2 \bar z^2),$$

where A and B are series in one variable with real coefficients, $A(0) = h_{22}$, $B(0) = |h_{40}|$. Moreover the coefficients of the series A and B are unique.

Proof. First we note that the resonant terms correspond to $k = l \pmod 4$ which is equivalent to the equality

$$k = l + 4j, \quad j \in \mathbb{Z},$$

which implies that $k + l$ are all even. It is convenient to illustrate distribution of the resonant terms using the diagram shown on Figure 3.

We will prove the proposition by induction transforming the Hamiltonian to the desired form order by order. On each step we will need to solve a homological equation which involves the operator L_2 defined by

$$L_2(\chi) = 2i \{\chi, h_2\} .$$

Before proceeding further let us study the action of this operator on homogeneous polynomials. Let us introduce $\mathcal{H}^4_m$ as the space of resonant terms of order $2m$. Unlike the previous section, where we used the real vector spaces, it is more convenient to assume that $\mathcal{H}^4_m$ consists of linear combination of resonant monomials

$$Q_{m,j} = z^{m+2j} \bar z^{m-2j}, \qquad -\left\lfloor \frac{m}{2} \right\rfloor \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

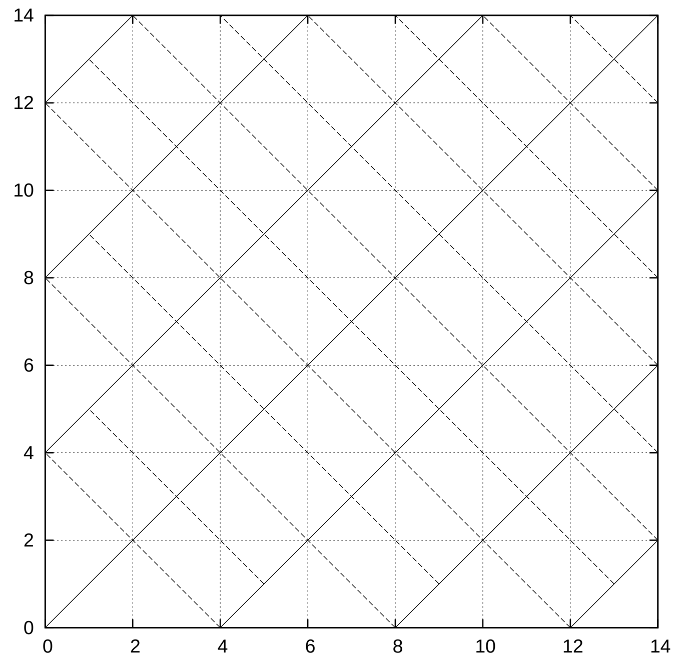

Figure 3. On the (k, l) plane, resonant terms in the Hamiltonian for $n = 4$ correspond to intersections of solid and dashed lines. The dashed lines connect the terms of equal δ-order

with complex coefficients. Consequently,

$$\dim \mathcal{H}_m^4 = 2\left\lfloor \frac{m}{2} \right\rfloor + 1\,. \tag{5.2}$$

Since $\lfloor\cdot\rfloor$ denotes the integer part of a number, the number of the resonant monomials of a given order form the sequence $3, 3, 5, 5, 7, 7, \ldots$ (see Figure 3).

Then real-valued polynomials form a (real) subspace in $\mathcal{H}_m^4$ (the coefficients in front of $Q_{m,\pm j}$ are mutually complex conjugate).

Taking into account (5.1) we obtain

$$\begin{aligned} L_2(\chi) &= 2i\,\{\chi, h_2\} \\ &= 4ia_0\left(z^2\bar{z}\frac{\partial\chi}{\partial z} - z\bar{z}^2\frac{\partial\chi}{\partial\bar{z}}\right) + 8ib_0\left(\bar{z}^3\frac{\partial\chi}{\partial z} - z^3\frac{\partial\chi}{\partial\bar{z}}\right). \end{aligned}$$

We see that if χ is a homogeneous polynomial of order p then $L_2(\chi)$ is a homogeneous polynomial of order $p+2$. Moreover L_2 maps resonant monomials into resonant ones. Then

$$L_2 : \mathcal{H}_m^4 \to \mathcal{H}_{m+1}^4.$$

Let us find a subspace complement to $L_2(\mathcal{H}_m^4)$ in $\mathcal{H}_{m+1}^4$.

A straightforward substitution into the definition of L_2 shows that

$$\begin{aligned} L_2(Q_{m,j}) &= 16ia_0 jQ_{m+1,j} \\ &\quad + 8ib_0(m+2j)Q_{m+1,j-1} - 8ib_0(m-2j)Q_{m+1,j+1}\,. \end{aligned}$$

Then L_2 is described by a tridiagonal matrix with coefficients s_{kj} given by

$$s_{j,j+1} = -8ib_0(m-2j), \qquad s_{j,j-1} = 8ib_0(m+2j), \qquad s_{j,j} = 16ia_0j\,. \tag{5.3}$$

We note that equation (5.2) gives us

$$\dim \mathcal{H}^4_m = \begin{cases} m\,, & m \text{ odd,} \\ m+1\,, & m \text{ even.} \end{cases}$$

So we have to treat two separate cases, namely m odd and m even.

Let us start by considering the case when m is odd. In this case

$$\dim \mathcal{H}^4_m = m$$

and

$$\dim \mathcal{H}^4_{m+1} = m+2\,.$$

We see that L_2 acts from a space of a lower dimension into a space of a higher dimension. Its matrix has $(m+2)$ rows and m columns. The non-zero elements are given by (5.3) where $-\frac{m-1}{2} \le j \le \frac{m-1}{2}$. For example in the case of $m = 5$ the matrix has the following structure

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 \\ x & x & 0 & 0 & 0 \\ x & x & x & 0 & 0 \\ 0 & x & 0 & x & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix}$$

where x occupies positions of non-zero elements. It is easy to see that since $b_0 \neq 0$ there is a $m \times m$ block with a non vanishing determinant (for example the first m rows form a lower diagonal matrix so its determinant is a straightforward product). We conclude that if m is odd

$$\operatorname{rank}(L_2) = m.$$

Since the rank is maximal the kernel is trivial:

$$\ker(L_2) = \mathbf{0}.$$

For m even we have

$$\dim \mathcal{H}^4_m = \dim \mathcal{H}^4_{m+1} = m+1\,.$$

Then L_2 is described by a square $(m+1) \times (m+1)$ tridiagonal matrix. The non-zero elements are given by (5.3) where $-\frac{m}{2} \le j \le \frac{m}{2}$. For example for $m = 4$ the matrix takes the form

$$\begin{bmatrix} x & x & 0 & 0 & 0 \\ x & x & x & 0 & 0 \\ 0 & x & 0 & x & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

where x occupies places of non-zero elements. To determine the rank of this matrix we first note that since $b_0 \neq 0$ the lower left block of size $m \times m$ is upper-diagonal with a non-zero determinant. Therefore rank $(L_2) \geq m$. On the other hand m is even and $h_2^{m/2}$ is a resonant homogeneous polynomial of order $2m$. It is in the kernel of L_2 because

$$L_2(h_2^{m/2}) = 2i\{h_2^{m/2}, h_2\} = 0\,.$$

Consequently rank $(L_2) < m+1$. We conclude that

$$\operatorname{rank}(L_2) = m$$

and the kernel is one dimensional,

$$\dim(\ker(L_2)) = 1\,,$$

and consists of elements proportional to $h_2^{m/2}$.

Summarising these results we see that

$$\text{co-dim}(L_2(\mathcal{H}_m^4)) = \begin{cases} 2, & \text{if } m \text{ is odd,} \\ 1, & \text{if } m \text{ is even.} \end{cases}$$

The real-valued polynomials form a (real) subspace of $\mathcal{H}_m^4$. The operator L_2 preserves real-valuedness. Consequently, the real co-dimensions of the image of L_2 restricted on the spaces of real-valued polynomials are given by the same formula.

Now we need an explicit description for the complements. A straightforward computation which uses the explicit formulae for the matrix coefficients (5.3) shows if m is odd the polynomials

$$d_1 z^{m+3}\bar{z}^{m-1} + d_0 z^{m+1}\bar{z}^{m+1} + d_1 z^{m-1}\bar{z}^{m+3}$$

with real d_0, d_1 do not have preimages under L_2. The case of even m is a bit more complicated. In this case the matrix of L_2 is square and its determinant vanishes. We replace the central column of this matrix (the one which corresponds to $j = 0$) by the vector $(0, \ldots, 0, d_0, 0, \ldots, 0)^T$ and check that the new matrix has a non-zero determinant. The computation of the determinant takes into account that the new matrix has block structure and each of the blocks is tridiagonal. Consequently, the added vector is linearly independent from the columns of the matrix of L_2 and belongs to the complement to its image. Therefore for m even

$$d_0 z^{m+1}\bar{z}^{m+1}$$

is not in the image of L_2.

We constructed two subspaces of dimensions two and one respectively which have trivial intersection with the image of L_2. Consequently, they provide the desired complements to $L_2(\mathcal{H}_m^4)$. We note that these complements are described by the same formulae as in the case $n \geq 5$.

Now we proceed to the proof of Proposition 5.1. Let χ_m be a real-valued homogeneous polynomial of order $2m$. Then

$$\widetilde{h} = h \circ \Phi_{\chi_m}^1 = \exp(L_{\chi_m})h = h + L_2(\chi_m) + O_{2m+4}\,,$$

where O_{2m+4} denotes a formal series without terms of orders lower than $2m+4$. We remind that L_2 increases the order of a homogeneous polynomial by 2. Therefore h and $\tilde{h}$ coincide up to the order $2m+1$ and

$$[\tilde{h}]_{m+1} = [h]_{m+1} + L_2(\chi_m)\,.$$

We choose χ_m in such a way that $[\tilde{h}]_{m+1}$ is in the complement to $L_2(\mathcal{H}^4_m) \subset \mathcal{H}^4_{m+1}$. Then replace m by $m+1$ and repeat the procedure.

The proof of the uniqueness uses essentially the same arguments as we used in the previous section. Suppose h can be transformed to two different simplified normal forms $\tilde{h}$ and $\tilde{h}'$ due to non-uniqueness of transformations to the normal form. Then there is a canonical transformation ϕ such that

$$\tilde{h} = \tilde{h}' \circ \phi\,.$$

Since the transformation ϕ is tangent to identity there is a formal real-valued Hamiltonian χ such that

$$\phi = \Phi^1_\chi\,.$$

Suppose that $2p$ is the lowest order of χ. Then $\tilde{h}$ and $\tilde{h}'$ coincide up to the order $2p+1$ and

$$[\tilde{h}]_{2p+2} = [\tilde{h}']_{2p+2} + L_2(\chi_{2p})\,.$$

Since both $[\tilde{h}]_{2p+2}$ and $[\tilde{h}']_{2p+2}$ are in the complement subspace to the image of L_2, we conclude that $L_2(\chi_{2p}) = 0$ and $[\tilde{h}]_{2p+2} = [\tilde{h}']_{2p+2}$.

Moreover either $\chi_{2p} = 0$ if p odd, or $\chi_{2p} = c\,h_2^{p/2}$ for some $c \in \mathbb{R}$ if p is even. Then the change of variables

$$\tilde{\phi} = \Phi^1_\chi \circ \Phi^1_{-c\tilde{h}^{p/2}}$$

also transforms $\tilde{h}'$ into $\tilde{h}$. It is easy to check that the corresponding Hamiltonian $\tilde{\chi}$ starts with order $p+2$.

Repeating the arguments inductively we see that $\tilde{h}$ and $\tilde{h}'$ coincide at all orders. Hence the simplified normal form is unique. □

Remark 5.2 *In the symplectic polar coordinates (I,φ) the normal form*

$$\widetilde{h}(z,\bar{z}) = z^2\bar{z}^2 a(z\bar{z}) + (z^4 + \bar{z}^4) b(z\bar{z}),$$

takes the form

$$H(I,\varphi) = I^2 A(I) + I^2 B(I)\cos(4\varphi).$$

We remind that $I = z\bar{z}/2$, $\varphi = \arg(z)$ or equivalently $z = \sqrt{2I}\,e^{i\varphi}$.

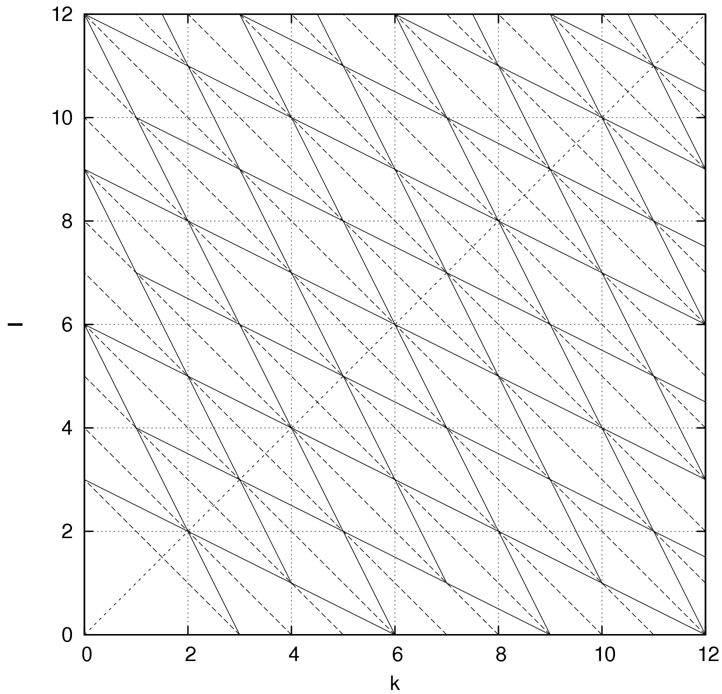

Figure 4. Resonant terms for $n = 3$ correspond to points of intersections of solid lines. Dashed lines connects terms of equal orders.

6. Third order resonance

This case is reduced to a resonant Hamiltonian of the form

$$h(z,\bar z) = \sum_{\substack{k+l\ge 3\\ k=l \pmod 3}} h_{kl} z^k \bar z^l \,. \tag{6.1}$$

Proposition 6.1 *If $h_{30} \neq 0$, there is a formal canonical change of variables which transforms $h(z,\bar z)$ into*

$$\widetilde{h}(z,\bar z) = z^3\bar z^3 A(z\bar z) + (z^3 + \bar z^3) B(z\bar z), \tag{6.2}$$

where A and B are series in one variable with real coefficients:

$$A(z\bar z) = \sum_{\substack{k\ge 0\\ k\ne 2 \pmod 3}} a_k z^k \bar z^k\,, \quad B(z\bar z) = \sum_{\substack{k\ge 0\\ k\ne 2 \pmod 3}} b_k z^k \bar z^k\,,$$

where $b_0 = |h_{30}|$. Moreover, the coefficients of the series A and B are unique.

Proof. Similarly to the previous section a rotation of the coordinates makes the coefficient of the leading order real and the Hamiltonian takes the form

$$h(z,\bar z) = b_0(z^3 + \bar z^3) + \sum_{\substack{k+l\ge 4\\ k=l \pmod 3}} h_{kl} z^k \bar z^l$$

where $b_0 = |h_{30}|$. We group together terms of the same order and define $\mathcal{H}^3_m$ to be the set of real-valued homogeneous resonant polynomials of order m. It is convenient to

m	$\dim \mathcal{H}^3_m$	$\dim \mathcal{H}^3_{m+1}$	$\dim \ker L_3$	co$-$dim Image L_3
$3k$	$k+1$	k	1	0
$3k+1$	k	$k+1$	0	1
$3k+2$	$k+1$	$k+2$	0	1

Table 1. Properties of the homological operator for $n = 3$.

represent the resonant terms using a diagram shown on Figure 4. It can be checked by induction that

$$\begin{aligned}\dim \mathcal{H}^3_{3k} &= k+1,\\ \dim \mathcal{H}^3_{3k+1} &= k,\\ \dim \mathcal{H}^3_{3k+2} &= k+1\,.\end{aligned}$$

The leading order of h is given by $h_3 = b_0(z^3 + \bar z^3)$, and the homological operator has the form

$$L_3(\chi) = 2i\{\chi, h_3\} = 6ib_0 \bar z^2 \frac{\partial \chi}{\partial z} - 6ib_0 z^2 \frac{\partial \chi}{\partial \bar z}\,. \tag{6.3}$$

This formula implies that $L(\mathcal{H}^3_m) \subset \mathcal{H}^3_{m+1}$. In order to study the normal form we need a description of the complement to the image. Uniqueness properties are related to the properties of the kernel. A standard result from Linear Algebra provides the following relation:

$$\text{co-dim}\,\text{Image}(L_3) = \dim \mathcal{H}^3_{m+1} - \dim \mathcal{H}^3_m + \dim \ker L_3\,.$$

Obviously $L_3(h_3^k) = 2i\{h_3^k, h^3\} = 0$ for any $k \in \mathbb{N}$ and therefore the kernel of L_3 restricted on $\mathcal{H}^3_{3k}$ is not trivial. The results of the study of L_3 are summarised in Table 1.

It is convenient to write a resonant monomial of order m in the form

$$Q_{mj} := z^{(m+3j)/2} \bar z^{(m-3j)/2}$$

where $-\left\lfloor \frac{m}{3} \right\rfloor \le j \le \left\lfloor \frac{m}{3} \right\rfloor$ and $j = m \pmod 2$. Therefore for a fixed m the index j changes with step 2. A direct substitution to the definition of L_3 shows

$$\begin{aligned}L_3(Q_{mj}) = 6ib_0 &\frac{m+3j}{2} z^{(m+3j)/2-1} \bar z^{(m-3j)/2+2}\\ &- 6ib_0 \frac{m-3j}{2} z^{(m+3j)/2+2} \bar z^{(m-3j)/2-1}\\ &= 6ib_0 \frac{m+3j}{2} Q_{m+1,j-1} - 6ib_0 \frac{m-3j}{2} Q_{m+1,j+1}\,.\end{aligned}$$

The action of L_3 is represented in the diagram shown on Figure 5. We see that for all m the matrix of L_3 is two diagonal. Analysing the cases of $m = 3k$, $m = 3k+1$ and $m = 3k+2$ separately, we see that since $b_0 \ne 0$ the rank of the matrix is maximal and equals to k, k and $k+1$ respectively. Consequently, if $m \ne 0 \pmod 3$ the kernel is empty, and if $m = 0 \pmod 3$ the kernel is one dimensional and generated by h_3^k.

We see that the image of L_3 completely covers $\mathcal{H}^3_p$ with $p = 1 \pmod 3$ and has one dimensional complement otherwise. Taking into account the structure of the matrix of

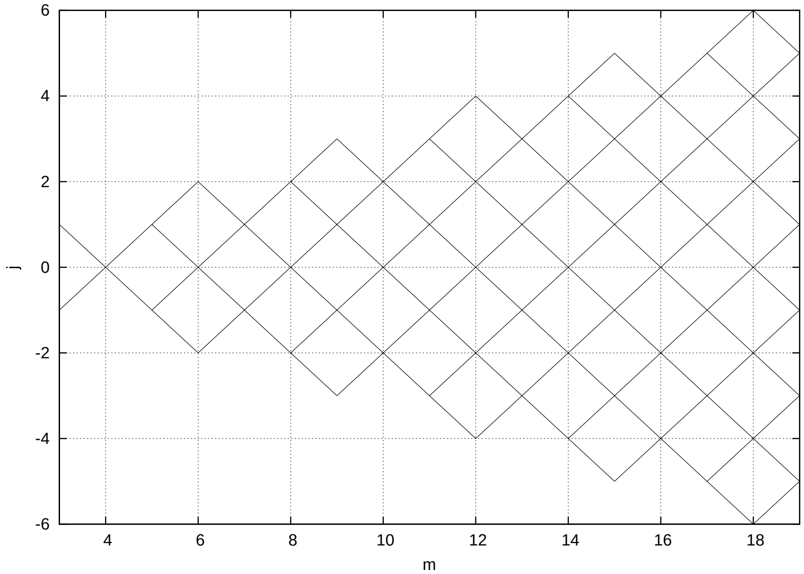

Figure 5. Action of the operator $L_3 : \mathcal{H}^3_m \to \mathcal{H}^3_{m+1}$ on monomials. Each monomial is connected by a line (or two lines) to its image.

L_3 we see that the complements are generate either by $Q_{p,0}$ if p is even, or by $Q_{p,1}+Q_{p,-1}$ if p is odd.

Now we follow the same strategy we used in the previous two sections. We construct inductively a sequence of substitutions:

$$\tilde{h} = \exp(L_{\chi_{p-1}})h = h + L_3(\chi_{p-1}) + O_{2p-3}$$

and choose χ_{p-1} in such a way that $[\tilde{h}]_p$ is in the complement space to $L_3(\mathcal{H}^3_{p-1}) \subset \mathcal{H}^3_p$. We see that $[\tilde{h}]_p = 0$ if $p = 1 \pmod 3$, and provided $p \neq 1 \pmod 3$ we get

$$[\tilde{h}]_p = a_k z^{k+3}\bar{z}^{k+3} \qquad \text{for } p = 2k+6\,,$$
$$[\tilde{h}]_p = b_k z^k \bar{z}^k (z^3 + \bar{z}^3) \qquad \text{for } p = 2k+3,$$

for some $a_k, b_k \in \mathbb{R}$. We have chosen k in such a way that $k = 0$ corresponds to the lowest non-zero order. Indeed, the lowest odd order in $\tilde{h}$ is obviously 3, and the lowest even order is 6 and not 4 because $4 = 1 \pmod 3$.

Repeating the argument inductively, we show that h can be transformed to the form (6.2). Uniqueness follows from the fact that the kernel of L_3 is generated by powers of h_3 only and we omit it since it repeats literally arguments from the proofs of Propositions 4.3 and 5.1. □

7. Quasi-resonant normal forms

In this section we prove Theorem 1.4. Instead of the individual map F_0 we consider an analytic family of area-preserving maps F_ε, which coincides with F_0 at $\varepsilon = 0$. Without loosing in generality we assume that for all ε the fixed point is at the origin. Then in the complex variables $(z,\bar z)$ the first component of F_ε can be written in the form of a Taylor series:

$$f_\varepsilon(z,\bar z)=\mu z+\sum_{\substack{k+l+j\ge 2\\ k,l,j\ge 0}} c_{klj} z^k \bar z^l \varepsilon^j\,,$$

where μ is the multiplier of F_0. We see that the series involves three variables $(z,\bar z,\varepsilon)$ instead of two variables $(z,\bar z)$. Moreover, the classical normal form theory implies that there is a formal change of variables which eliminates all non-resonant terms from this sum. We will assume that this change of variables has been done.

As in the Theorem 3.1 it can be shown that there exists a unique real-valued formal Hamiltonian

$$h(z,\bar z;\varepsilon)=\sum_{\substack{k+l+j\ge 3\\ k,l,j\ge 0}} h_{klj} z^k \bar z^l \varepsilon^j \tag{7.1}$$

such that

$$f_\varepsilon=\mu\exp(L_{h_\varepsilon})z.$$

We note that the sum in (7.1) contains only resonant terms $k=l \pmod n$ (assuming $\mu^n=1$). The proof of this statement is similar to the proof provided in Section 3 for the case of an individual map. Of course, one should consider homogeneous polynomials in three variables.

Proposition 7.1 *If $h_{n00}\ne 0$, there is a formal canonical change of variables which transforms $h(z,\bar z;\varepsilon)$ into*

$$\widetilde h(z,\bar z;\varepsilon)=z\bar z A(z\bar z;\varepsilon)+(z^n+\bar z^n)B(z\bar z;\varepsilon), \tag{7.2}$$

where A and B are series in two variables with real coefficients:

- *if $n\ge 4$ and $h_{220}h_{n00}\ne 0$*

$$A(z\bar z,\varepsilon)=\sum_{km\ge 0} a_{km} z^k \bar z^k \varepsilon^m,\quad B(z\bar z,\varepsilon)=\sum_{km\ge 0} b_{km} z^k \bar z^k \varepsilon^m\,, \tag{7.3}$$

  where $a_{00}=0$,

- *if $n=3$ and $h_{300}\ne 0$*

$$A(I,\varepsilon)=\sum_{\substack{km\ge 0\\ k\ne 1 \pmod 3}} a_{km} z^k \bar z^k \varepsilon^m\,,\qquad a_{00}=a_{10}=0\,, \tag{7.4}$$

$$B(I,\varepsilon)=\sum_{\substack{km\ge 0\\ k\ne 2 \pmod 3}} b_{km} z^k \bar z^k \varepsilon^m\,, \tag{7.5}$$

where $b_{00}=|h_{n00}|$. Moreover the coefficients of the series A and B are unique.

Proof. The scheme of the proof is similar to the previous sections. The unique normal form is constructed by induction: we use a sequence of canonical transformations to eliminate as many resonant terms as possible. As in the previous sections we first use the rotation $z \mapsto e^{-i\arg(h_{n00})/n}\, z$ which transforms

$$h_{n00} z^n + h_{0n0} \bar{z}^n \mapsto b_0(z^n + \bar{z}^n),$$

where $b_{00} = |h_{n00}|$.

First we consider the case of $n \geq 4$. The terms in (7.1) can be grouped in the following way:

$$h(z, \bar{z}; \varepsilon) = \sum_{s\geq 2} \sum_{j=0}^{s-1} \varepsilon^j h_{s-j,j}$$

where $h_{s-j,j} \in \mathcal{H}^n_{s-j}$. We note that after the rotation the terms with $s = 2$ already have the desired form:

$$h_{2,0} = b_{00}(z^n + \bar{z}^n) \qquad \text{and} \qquad h_{1,1} = h_{111} z\bar{z}\,.$$

and we simply let $a_{11} = h_{111}$. We will simplify the terms of the formal series in the following order: for each fixed s starting with $s = 3$, we will run j from 0 to $s - 1$. Following this order, we perform a sequence of canonical coordinate changes generated by an auxiliary Hamiltonian $\chi_{p-k,k} \in \mathcal{H}^n_{p-k}$:

$$(z, \bar{z}) \mapsto \Phi^{\varepsilon^k}_{\chi_{p-k,k}}(z, \bar{z})\,.$$

This is a canonical change of variables and the Hamiltonian h is transformed into

$$\tilde{h} = \exp\left(\varepsilon^k \chi_{p-k,k}\right) h\,.$$

The function $\chi_{p-k,k}$ will be chosen to normalize the term $\varepsilon^k h_{p-k+1,k}$. Writing Lie series for the transformed Hamiltonian we get

$$\tilde{h} = h + \varepsilon^k L_{\chi_{p-k,k}} h + \sum_{l\geq 2} \varepsilon^{kl} \frac{1}{l!} L^l_{\chi_{p-k,k}} h\,.$$

It is easy to check that

$$\tilde{h}_{s-j,j} = h_{s-j,j}$$

for $s \leq p$ $(0 \leq j \leq s-1)$ and for $s = p+1, j < k$. For $s = p+1, j = k$ we get

$$\tilde{h}_{p-k+1,k} = h_{p-k+1,k} + \left[L_{\chi_{p-k,k}} h_{2,0}\right]_{p-k+1}. \tag{7.6}$$

Since $h_{2,0} = h_{2,2,0} z^2 \bar{z}^2 + b_{00}(z^n + \bar{z}^n)$ agrees with h_2 in (4.5) for $n > 4$ or (5.1) for $n = 4$ we can rewrite the formula using the homological operator L_{p-k} ($L_{p-k} = L_2$ for $n = 4$):

$$\tilde{h}_{p-k+1,k} = h_{p-k+1,k} + L_{p-k} \chi_{p-k,k}\,.$$

The explicit description for the complements of the homological operators was provided in Sections 4 and 5 for $n \geq 5$ and $n = 4$ respectively. So there exists $\chi_{p-k,k} \in \mathcal{H}^n_{p-k}$ such that $\tilde{h}_{p-k+1,k}$ takes the form (4.8) if $p - k$ is odd and (4.9) if $p - k$ is even.

In the case of the third order resonance the Hamiltonian (7.1) can be written as

$$h(z, \bar{z}; \varepsilon) = \sum_{s\geq 3} \sum_{j=0}^{s-1} \varepsilon^j h_{s-j,j}\,,$$

where $h_{s-j,j} \in \mathcal{H}^3_{s-j}$. Then we continue in the same way as in the case of $n \geq 4$ but Equation (7.6) is replaced by

$$\tilde{h}_{p-k+1,k} = h_{p-k+1,k} + L_{\chi_{p-k,k}} h_{3,0}.$$

Since $h_{3,0} = h_3$ then $L_{\chi_{p-k,k}} h_{3,0} = L_3(\chi_{p-k,k})$, where the homological operator L_3 is defined by (6.3). Using the results of Section 6 we get

$$\tilde{h}_{p-k,k} = 0 \quad \text{for } p-k = 1 \pmod 3$$

and if $p-k \neq 1 \pmod 3$

$$[\tilde{h}]_{p-k,k} = a_{l,k} z^{l+1} \bar{z}^{l+1} \qquad \text{for } p-k = 2l+6;\ a_{0,0} = a_{1,0} = 0\,,$$
$$[\tilde{h}]_{p-k,k} = b_{l,k} z^l \bar{z}^l (z^3 + \bar{z}^3) \qquad \text{for } p-k = 2l+3.$$

The proof of the uniqueness is very similar to the previous sections. Suppose h can be transformed into two different simplified normal forms $\tilde{h}$ and $\tilde{h}'$. The formal Hamiltonian χ in (4.10) now takes the form:

$$\chi = \sum_{s \geq p} \sum_{j=0}^{s-1} \varepsilon^j \chi_{s-j,j}$$

and the lowest order is given by the first non-zero term $\varepsilon^k \chi_{p-k,k}$. Then

$$\tilde{h}_{s-j,j} = \tilde{h}'_{s-j,j} \text{ for } s \leq p \ (0 \leq j \leq s-1) \text{ and for } s = p+1, j < k$$

and

$$\tilde{h}'_{p-k+1,k} = \tilde{h}_{p-k+1,k} + L_{p-k} \chi_{p-k,k}\,,$$

where L_{p-k} is the homological operator ($L_{p-k} = L_2$ for $n = 4$ and $L_{p-k} = L_3$ for $n = 3$). As it was shown in Sections 4 and 5 the knowledge of the kernel of the homological operator allows us to show the existence of $\tilde{\chi}$ which conjugates $\tilde{h}$ and $\tilde{h}'$ but starts at least from the next order in comparison with χ. Consequently $\tilde{h}'_{p-k+1,k} = \tilde{h}_{p-k+1,k}$ and we conclude $\tilde{h} = \tilde{h}'$ by induction.

□

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